External Appendix to Accompany "Good Volatility, Bad Volatility, and Option Pricing"*

Bruno Feunou
Financial Markets Department
Bank of Canada

Cédric Okou
ESG
UQAM

September 15, 2017

Abstract

This External Appendix contains explicit expressions for the moment generating function, additional empirical model properties, along with risk-neutralization formulas.

*Please address correspondence to Bruno Feunou at feun@bankofcanada.ca. Cédric Okou is at okou.cedric@uqam.ca.

A Physical moment generating function

We derive the closed-form moment generating function (MGF) for the one-factor skew affine realized variance (SARV) model under the physical probability measure. Note that our generalized skew affine realized variance (GSARV) model is a straightforward two-factor extension or a convolution of two SARV models. The dynamics of log-returns are modelled as

$$R_{t+1} = \bar{r} + (\lambda - \xi) h_t + z_{t+1},$$

with

$$z_{t+1} = \sqrt{h_t} \eta_{t+1},$$

$$\eta_{t+1} = -\sqrt{\frac{\omega}{2h_t}} \left[\left(\varepsilon_{t+1}^{(1)} - \sqrt{\frac{h_t - \omega}{2\omega}} \right)^2 - \left(\frac{h_t + \omega}{2\omega} \right) \right],$$

and $\varepsilon_{t+1}^{(1)}$ are *i.i.d.* $N(0,1)$.

From $h_0 \geq \omega$, we have

$$\begin{aligned} h_{t+1} - \omega &= \varpi + \beta (h_t - \omega) + \alpha \left(\varepsilon_{t+1}^{(2)} - \gamma \sqrt{h_t - \omega} \right)^2, \\ RV_{t+1} &= h_t + \sigma \left[\left(\varepsilon_{t+1}^{(2)} - \gamma \sqrt{h_t - \omega} \right)^2 - \left(1 + \gamma^2 (h_t - \omega) \right) \right], \end{aligned}$$

where

$$E\left[\varepsilon_{t+1}^{(1)} \varepsilon_{t+1}^{(2)} \right] = \rho.$$

In addition to λ, the other constant terms in the conditional expectation of returns have the following expressions

$$\begin{aligned} \bar{r} &= r_f + \frac{1}{2} \left[\ln \left(1 + \sqrt{2\omega} \right) - \frac{\omega + \sqrt{2\omega}}{\left(1 + \sqrt{2\omega} \right)} \right], \\ \xi &= \frac{1}{2 \left(1 + \sqrt{2\omega} \right)}. \end{aligned}$$

Thus, the one-step-ahead joint conditional MGF of return and variance writes

$$\begin{aligned} & E_t\left[\exp\left(uR_{t+1}+vh_{t+1}\right)\right] \\ = & \exp\left(u\left(\bar{r}+(\lambda-\xi)\,h_t\right)+u\sqrt{\frac{\omega}{2}}\left(\frac{h_t+\omega}{2\omega}\right)+v\left(\omega+\varpi+\beta\left(h_t-\omega\right)\right)\right) \\ & \times E_t\left[\exp\left(-u\sqrt{\frac{\omega}{2}}\left(\varepsilon_{t+1}^{(1)}-\sqrt{\frac{h_t-\omega}{2\omega}}\right)^2+v\alpha\left(\varepsilon_{t+1}^{(2)}-\gamma\sqrt{h_t-\omega}\right)^2\right)\right]. \end{aligned}$$

Using the Cholesky representation, we can rewrite

$$\varepsilon_{t+1}^{(1)}=\rho\varepsilon_{t+1}^{(2)}+\sqrt{1-\rho^2}\varepsilon_{t+1}^{(3)},$$

where $E\left[\varepsilon_{t+1}^{(2)}\varepsilon_{t+1}^{(3)}\right]=0$. Moreover, by exploiting the identity

$$\log E\left[\exp\left(a(z+b)^2\right)\right]=\frac{ab^2}{1-2a}-\frac{1}{2}\log(1-2a),$$

for z distributed as a standard normal random variable, one can check that

$$\begin{aligned} & \log E_t\left[\exp\left(-u\sqrt{\frac{\omega}{2}}\left(\varepsilon_{t+1}^{(1)}-\sqrt{\frac{h_t-\omega}{2\omega}}\right)^2+v\alpha\left(\varepsilon_{t+1}^{(2)}-\gamma\sqrt{h_t-\omega}\right)^2\right)\right] \\ = & -\frac{1}{2}\log\left(1-2v\alpha+\sqrt{2\omega}u\left(1-2v\alpha\left(1-\rho^2\right)\right)\right) \\ & +\left(\frac{\gamma^2v\alpha+2\gamma^2v\alpha\sqrt{\frac{\omega}{2}}u-2u\rho\gamma v\alpha-(1-2v\alpha)\,u\frac{1}{2\omega}\sqrt{\frac{\omega}{2}}}{(1-2v\alpha)\left(1+2u\left(1-\rho^2\right)\sqrt{\frac{\omega}{2}}\right)+2\sqrt{\frac{\omega}{2}}\rho^2u}\right)(h_t-\omega). \end{aligned}$$

Hence,

$$\begin{aligned} & E_t\left[\exp\left(uR_{t+1}+vh_{t+1}\right)\right] \\ = & \exp\left(\begin{array}{c} -\frac{1}{2}\log\left(1-2v\alpha+\sqrt{2\omega}u\left(1-2v\alpha\left(1-\rho^2\right)\right)\right) \\ +u\left(\bar{r}+\sqrt{\frac{\omega}{2}}+(\lambda-\xi)\,h_t\right)+v\left(\omega+\varpi+\beta\left(h_t-\omega\right)\right) \end{array}\right) \\ & \times\exp\left(\frac{\frac{1}{2}\left(1-2v\alpha\left(1-\rho^2\right)\right)u^2+\gamma v\alpha\left(\gamma+2\left(\gamma\sqrt{\frac{\omega}{2}}-\rho\right)u\right)}{1-2v\alpha+\sqrt{2\omega}u\left(1-2v\alpha\left(1-\rho^2\right)\right)}(h_t-\omega)\right). \end{aligned}$$

To ease notation, define

$$\begin{aligned}\zeta(u,v) &= -\frac{1}{2}\log\left(1-2v\alpha+\sqrt{2\omega}u\left(1-2v\alpha\left(1-\rho^2\right)\right)\right),\\ \psi(u,v) &= \frac{\frac{1}{2}\left(1-2v\alpha\left(1-\rho^2\right)\right)u^2+\gamma v\alpha\left(\gamma+2\left(\gamma\sqrt{\frac{\omega}{2}}-\rho\right)u\right)}{1-2v\alpha+\sqrt{2\omega}u\left(1-2v\alpha\left(1-\rho^2\right)\right)}.\end{aligned}$$

Accordingly, the one-step-ahead joint conditional MGF of return and variance can be reexpressed in a compact form as

$$E_t\left[\exp\left(uR_{t+1}+vh_{t+1}\right)\right]\equiv\exp\left(A(u,v)h_t+B(u,v)\right),$$

with

$$\begin{aligned}B(u,v) &= \zeta(u,v)+u\left(\bar{r}+\sqrt{\frac{\omega}{2}}\right)+v\left(\omega+\varpi-\omega\beta\right)-\omega\psi(u,v)\\ A(u,v) &= \psi(u,v)+u(\lambda-\xi)+v\beta\end{aligned}$$

Thus, the physical one-step-ahead moment generating function (MGF) is exponentially affine.

Interestingly, the MGF of the general two-factor model can be deduced as

$$E_t\left[\exp\left(\nu R_{t+1}+v_u h_{u,t+1}+v_d h_{d,t+1}\right)\right]\equiv\exp\left(A_u(\nu,v_u)h_{u,t}+A_d(\nu,v_d)h_{d,t}+B(\nu,v_u,v_d)\right),$$

with

$$\begin{aligned}A_u(\nu,v) &= \psi_u(-\nu,v)+u\left(\lambda_u-\xi_u\right)+v\beta_u,\\ A_d(\nu,v) &= \psi_d(\nu,v)+u\left(\lambda_d-\xi_d\right)+v\beta_d,\\ B(\nu,v_u,v_d) &= u\begin{pmatrix} r_f+\frac{1}{2}\left[\ln\left(1-\sqrt{2\omega_u}\right)-\frac{\omega_u-\sqrt{2\omega_u}}{1-\sqrt{2\omega_u}}\right]-\sqrt{\frac{\omega_u}{2}}\\ +\frac{1}{2}\left[\ln\left(1+\sqrt{2\omega_d}\right)-\frac{\omega_d+\sqrt{2\omega_d}}{1+\sqrt{2\omega_d}}\right]+\sqrt{\frac{\omega_d}{2}}\end{pmatrix}\\ &\quad+\zeta_u(-\nu,v_u)+\zeta_d(\nu,v_d)+v_u\left(\omega_u+\varpi_u-\omega_u\beta_u\right)\\ &\quad+v_d\left(\omega_d+\varpi_d-\omega_d\beta_d\right)-\omega_u\psi(-\nu,v_u)-\omega_d\psi(\nu,v_d).\end{aligned}$$

We conjecture that the multi-step MGF is also of the affine form. First, define

$$\begin{aligned}\Psi_{t,t+M}(u) &= E_t[\exp(u\sum_{j=1}^{M} R_{t+j})] \\ &= \exp(C(u,M)'h_t + D(u,M)).\end{aligned}$$

Taking advantage of the affine structure of the model we can compute

$$\begin{aligned}\Psi_{t,t+M+1}(u) &= E_t[\exp(u\sum_{j=1}^{M} R_{t+j})] = E_t[E_{t+1}[\exp(u\sum_{j=1}^{M} R_{t+j})]] \\ &= E_t[\exp(uR_{t+1})E_{t+1}[\exp(u\sum_{j=2}^{M} R_{t+j})]] \\ &= E_t[\exp(uR_{t+1} + C(u,M)'h_{t+1} + D(u,M))] \\ &= \exp(A(u,C(u,M))'h_t + B(u,C(u,M)) + D(u,M)),\end{aligned}$$

which yields the following recursive relationship

$$\begin{aligned}C(u,M+1) &= A(u,C(u,M)), \\ D(u,M+1) &= B(u,C(u,M)) + D(u,M),\end{aligned}$$

including the following initial conditions

$$\begin{aligned}C(u,1) &= A(u,\mathbf{0}), \\ D(u,1) &= B(u,\mathbf{0}),\end{aligned}$$

where A and C are 2-by-1 vector-valued functions.

B Additional model properties for physical estimation

We investigate additional empirical properties of the various models. Namely, we explore the ability of each specification to generate realistic historical conditional volatility paths, consistent volatility of variance dynamics, and coherent conditional correlation patterns between returns and variances.

For each model in Figure A1, we plot the daily conditional volatility computed as the square root of h_{t+1}. Clearly, the GARCH model does a poor job in replicating extreme volatility episodes in our sample window, as compared to the other models. All models (except GARCH) appear to track the market spot volatility in a similar way. However, the one-factor ARV specification tends to exhibit slightly stronger spikes than two-factor CGSARV and GSARV models, probably because two-factor models are not directly maximized on RV, but rather on upside and downside components of RV.

The time paths of upside and downside conditional volatilities from two-factor models are presented in the left and right columns of Figure A2, respectively. Two-factor CGSARV, and GSARV models produce nearly identical temporal patterns for conditional semi volatilities.

The dynamics of model-based conditional volatility of variance in Figure A3 support the previous observations. Recall that the conditional volatility of variance is calculated as the square root of

$$
\begin{aligned}
Var_t(h_{t+1}) &= Var_t(h_{u,t+1}) + Var_t(h_{d,t+1}), \\
\text{where} & \\
Var_t(h_{j,t+1}) &= 2\alpha_j^2 \left(1 + 2\gamma_j^2 \left(h_{j,t} - \omega_j\right)\right), \text{ for } j = u, d.
\end{aligned}
$$

Except for the GARCH model that entails a relatively low and smooth conditional volatility of variance, all models deliver high (resp. low) conditional variability in variance when volatility is high (resp. low), consistent with observed empirical regularities. Moreover, two-factor specifications generate similar time series of conditional volatilities of upside and downside variances, as shown in Figure A4.

Figure A5 plots the time series of conditional correlations between return and variance, which are computed using

$$
\begin{aligned}
Corr_t(R_{t+1}, h_{t+1}) &= \frac{cov_t\left(R_{t+1}, h_{u,t+1}\right) + cov_t\left(R_{t+1}, h_{d,t+1}\right)}{\sqrt{h_{t+1} Var_t(h_{t+1})}}, \\
\text{where,} & \\
cov_t\left(R_{t+1}, h_{j,t+1}\right) &= \left(\sqrt{2\omega_j}\rho_j^2\alpha_j + 2\left(h_{j,t} - \omega_j\right)\rho_j\alpha_j\gamma_j\right)\left(\mathbb{I}_{\{j=u\}} - \mathbb{I}_{\{j=d\}}\right), \text{ for } j = u, d.
\end{aligned}
$$

Models under consideration display important differences in this regard. One-factor models imply conditional correlation values that are all negative. Specifically, the conditional correlation moves close to its lower bound of -1 for the GARCH model, and is constant at about -0.1 for the ARV model. By contrast, the conditional correlation alternates between negative and positive values in the CGSARV and GSARV specifications. Moreover, Figure A6 illustrates that these last two models yield positive (resp. negative) conditional correlations of return and upside (resp. downside) variance, consistent with the observed empirical evidence. These results lends additional credibility to the proposed GSARV model and underscores its ability to deliver option prices that closely reflect the real-world empirical regularities.

C Risk Neutralization

In this appendix, we derive the risk-neutralization formulas for the general model. We assume an exponential pricing kernel of the following form

$$M_{t+1} = M_{t+1}^{(u)} M_{t+1}^{(d)},$$

where

$$M_{t+1}^{(j)} = \frac{\exp\left(\nu_{1t}^{(1j)} \varepsilon_{j,t+1}^{(1)} + \nu_{2t}^{(1j)} \left(\varepsilon_{j,t+1}^{(1)}\right)^2 + \nu_{1t}^{(2j)} \varepsilon_{j,t+1}^{(2)} + \nu_{2t}^{(2j)} \left(\varepsilon_{j,t+1}^{(2)}\right)^2\right)}{E_t\left[\exp\left(\nu_{1t}^{(1j)} \varepsilon_{j,t+1}^{(1)} + \nu_{2t}^{(1j)} \left(\varepsilon_{j,t+1}^{(1j)}\right)^2 + \nu_{1t}^{(2j)} \varepsilon_{j,t+1}^{(2j)} + \nu_{2t}^{(2j)} \left(\varepsilon_{j,t+1}^{(2j)}\right)^2\right)\right]}, \text{ for } j = u, d.$$

We need to impose the no-arbitrage condition

$$E_t^Q \left[\exp\left(R_{t+1}\right)\right] \equiv E_t \left[M_{t+1} \exp\left(R_{t+1}\right)\right] = \exp\left(r_f\right).$$

One can show that

$$\begin{aligned} & E_t\left[\exp\left(\nu_{1t}^{(1)} \varepsilon_{j,t+1}^{(1)} + \nu_{2t}^{(1)} \left(\varepsilon_{j,t+1}^{(1)}\right)^2 + \nu_{1t}^{(2)} \varepsilon_{j,t+1}^{(2)} + \nu_{2t}^{(2)} \left(\varepsilon_{j,t+1}^{(2)}\right)^2\right)\right] \\ = \; & \exp\left(\begin{array}{c} -\frac{1}{2}\ln\left(1 - 2\nu_{2t}^{(1)} - 2\nu_{2t}^{(2)} + 4\left(1-\rho^2\right)\nu_{2t}^{(1)}\nu_{2t}^{(2)}\right) \\ + \frac{\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(2)}\right)\left(\nu_{1t}^{(1)}\right)^2 + \left(1-2\left(1-\rho^2\right)\nu_{2t}^{(1)}\right)\left(\nu_{1t}^{(2)}\right)^2 + 2\rho\nu_{1t}^{(1)}\nu_{1t}^{(2)}}{2\left(1-2\nu_{2t}^{(1)} - 2\nu_{2t}^{(2)} + 4\left(1-\rho^2\right)\nu_{2t}^{(1)}\nu_{2t}^{(2)}\right)} \end{array}\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
E_t^Q\left[\exp\left(u\varepsilon_{j,t+1}^{(1)}+v\varepsilon_{j,t+1}^{(2)}\right)\right] &= E_t\left[M_{t+1}\exp\left(u\varepsilon_{j,t+1}^{(1)}+v\varepsilon_{j,t+1}^{(2)}\right)\right] \\
&= \exp\left(\frac{\begin{array}{c}\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(2j)}\right)u^2+2\left[\nu_{1t}^{(1j)}\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(2j)}\right)+\rho\nu_{1t}^{(2j)}\right]u \\ +\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(1j)}\right)v^2+2\left[\nu_{1t}^{(2j)}\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(1j)}\right)+\rho\nu_{1t}^{(1j)}\right]v+2\rho uv\end{array}}{2\left(1-2\nu_{2t}^{(1j)}-2\nu_{2t}^{(2j)}+4\left(1-\rho^2\right)\nu_{2t}^{(1j)}\nu_{2t}^{(2j)}\right)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_{j,t+1}^{(1)} &\sim^Q N\left(\frac{\nu_{1t}^{(1j)}\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(2j)}\right)+\rho\nu_{1t}^{(2j)}}{1-2\nu_{2t}^{(1j)}-2\nu_{2t}^{(2j)}+4\left(1-\rho^2\right)\nu_{2t}^{(1j)}\nu_{2t}^{(2j)}},\frac{1-2\left(1-\rho^2\right)\nu_{2t}^{(2j)}}{1-2\nu_{2t}^{(1j)}-2\nu_{2t}^{(2j)}+4\left(1-\rho^2\right)\nu_{2t}^{(1j)}\nu_{2t}^{(2j)}}\right), \\
\varepsilon_{j,t+1}^{(2)} &\sim^Q N\left(\frac{\nu_{1t}^{(2j)}\left(1-2\left(1-\rho^2\right)\nu_{2t}^{(1j)}\right)+\rho\nu_{1t}^{(1j)}}{1-2\nu_{2t}^{(1j)}-2\nu_{2t}^{(2j)}+4\left(1-\rho^2\right)\nu_{2t}^{(1j)}\nu_{2t}^{(2j)}},\frac{1-2\left(1-\rho^2\right)\nu_{2t}^{(1j)}}{1-2\nu_{2t}^{(1j)}-2\nu_{2t}^{(2j)}+4\left(1-\rho^2\right)\nu_{2t}^{(1j)}\nu_{2t}^{(2j)}}\right), \\
\rho_{jt}^Q &\equiv corr_t^Q\left(\varepsilon_{j,t+1}^{(1)},\varepsilon_{j,t+1}^{(2)}\right)=\frac{\rho_j}{\sqrt{1-2\left(1-\rho^2\right)\nu_{2t}^{(2j)}}\sqrt{1-2\left(1-\rho^2\right)\nu_{2t}^{(1j)}}}.
\end{aligned}
$$

We will only consider constant $\nu_{2t}^{(1j)}$ and $\nu_{2t}^{(2j)}$. We posit

$$
\begin{aligned}
\kappa_1^j &= \frac{1-2\nu_2^{(1j)}-2\nu_2^{(2j)}+4\left(1-\rho_j^2\right)\nu_2^{(1j)}\nu_2^{(2j)}}{1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}}, \\
\kappa_2^j &= \frac{1-2\nu_2^{(1j)}-2\nu_2^{(2j)}+4\left(1-\rho_j^2\right)\nu_2^{(1j)}\nu_2^{(2j)}}{1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}},
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{j,t+1}^{(*1)} &= \sqrt{\kappa_1^j}\left(\varepsilon_{j,t+1}^{(1)}-\frac{\nu_{1t}^{(1j)}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)+\rho_j\nu_{1t}^{(2j)}}{\kappa_1^j\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)}\right), \\
\varepsilon_{j,t+1}^{(*2)} &= \sqrt{\kappa_2^j}\left(\varepsilon_{j,t+1}^{(2)}-\frac{\nu_{1t}^{(2j)}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}\right)+\rho_j\nu_{1t}^{(1j)}}{\kappa_2^j\left(1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}\right)}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\varepsilon_{j,t+1}^{(*1)} &\sim^{Q} N(0,1) \\
\varepsilon_{j,t+1}^{(*2)} &\sim^{Q} N(0,1) \\
E_t^Q\left[\varepsilon_{j,t+1}^{(*1)}\varepsilon_{j,t+1}^{(*2)}\right] &= \rho_j^Q = \frac{\rho_j}{\sqrt{1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}}\sqrt{1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}}}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
R_{t+1} &= \bar{r} + (\lambda_d - \xi_d)\,h_{dt} - \sqrt{\frac{\omega_d}{2}}\left[\left(\varepsilon_{d,t+1}^{(1)} - \sqrt{\frac{h_{dt}-\omega_d}{2\omega_d}}\right)^2 - \left(\frac{h_{dt}+\omega_d}{2\omega_d}\right)\right] \\
&\quad + (\lambda_u - \xi_u)\,h_{ut} + \sqrt{\frac{\omega_u}{2}}\left[\left(\varepsilon_{u,t+1}^{(1)} - \sqrt{\frac{h_{ut}-\omega_u}{2\omega_u}}\right)^2 - \left(\frac{h_{ut}+\omega_u}{2\omega_u}\right)\right] \\
&= \bar{r} + (\lambda_d - \xi_d)\,h_{dt} - \sqrt{\frac{\omega_d}{2\left(\kappa_1^d\right)^2}}\left[\begin{array}{c}\left(\begin{array}{c}\varepsilon_{d,t+1}^{(*1)} + \frac{\nu_{1t}^{(1d)}\left(1-2\left(1-\rho_d^2\right)\nu_2^{(2d)}\right)+\rho_d\nu_{1t}^{(2d)}}{\sqrt{\kappa_1^d}\left(1-2\left(1-\rho_d^2\right)\nu_2^{(2d)}\right)} \\ -\sqrt{\frac{\kappa_1^d\left(h_{dt}-\omega_d\right)}{2\omega_d}}\end{array}\right)^2 - \\ \kappa_1^d\left(\frac{h_{dt}+\omega_d}{2\omega_d}\right)\end{array}\right] \\
&\quad + (\lambda_u - \xi)\,h_t + \sqrt{\frac{\omega_u}{2\left(\kappa_1^u\right)^2}}\left[\begin{array}{c}\left(\begin{array}{c}\varepsilon_{u,t+1}^{(*1)} + \frac{\nu_{1t}^{(1u)}\left(1-2\left(1-\rho_u^2\right)\nu_2^{(2u)}\right)+\rho_u\nu_{1t}^{(2u)}}{\sqrt{\kappa_1^u}\left(1-2(1-\rho_u^2)\nu_2^{(2u)}\right)} \\ -\sqrt{\frac{\kappa_1^u\left(h_{ut}-\omega_u\right)}{2\omega_u}}\end{array}\right)^2 \\ -\kappa_1^u\left(\frac{h_{ut}+\omega_u}{2\omega_u}\right)\end{array}\right],
\end{aligned}
$$

It follows that

$$
h_t^Q \equiv Var_t^Q\left[R_{t+1}\right] = h_{ut}^Q + h_{dt}^Q,
$$

where

$$
h_{jt}^Q = \frac{\omega_j}{\left(\kappa_1^j\right)^2}\left[1 + 2\left(\frac{\nu_{1t}^{(1j)}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)+\rho_j\nu_{1t}^{(2j)}}{\sqrt{\kappa_1^j}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)} - \sqrt{\frac{\kappa_1^j\left(h_{jt}-\omega_j\right)}{2\omega_j}}\right)^2\right].
$$

To preserve the same structure, we also want to have

$$
\begin{aligned}
\sqrt{\frac{\omega_j}{2\left(\kappa_1^j\right)^2}} &\equiv \sqrt{\frac{\omega_j^Q}{2}} \\
\frac{\nu_{1t}^{(1j)}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)+\rho_j\nu_{1t}^{(2j)}}{\sqrt{\kappa_1^j}\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right)} - \sqrt{\frac{\kappa_1^j\left(h_{jt}-\omega_j\right)}{2\omega_j}} &= -\sqrt{\frac{h_{jt}^Q-\omega_j^Q}{2\omega_j^Q}},
\end{aligned}
$$

which implies that

$$
\omega_j^Q = \frac{\omega_j}{\left(\kappa_1^j\right)^2}.
$$

Building on the above results, the log return dynamics under the risk-neutral distribution writes

$$
\begin{aligned}
R_{t+1} &= \bar{r} + \left(\lambda_d-\xi_d\right)h_{dt} - \sqrt{\frac{\omega_d^Q}{2}}\left[\left(\varepsilon_{d,t+1}^{(*1)} - \sqrt{\frac{h_{dt}^Q-\omega_d^Q}{2\omega_d^Q}}\right)^2 - \kappa_1^d\left(\frac{h_{dt}+\omega_d}{2\omega_d}\right)\right] \\
&\quad + \left(\lambda_u-\xi_u\right)h_{ut} + \sqrt{\frac{\omega_u^Q}{2}}\left[\left(\varepsilon_{u,t+1}^{(*1)} - \sqrt{\frac{h_{ut}^Q-\omega_u^Q}{2\omega_u^Q}}\right)^2 - \kappa_1^u\left(\frac{h_{ut}+\omega_u}{2\omega_u}\right)\right] \\
&= \bar{r} + \left(\lambda_d-\xi_d\right)h_{dt} - \sqrt{\frac{\omega_d^Q}{2}}\left[\left(\frac{h_{dt}^Q+\omega_d^Q}{2\omega^{d^Q}}\right) - \kappa_1^d\left(\frac{h_{dt}+\omega_d}{2\omega_d}\right)\right] \\
&\quad - \sqrt{\frac{\omega_d^Q}{2}}\left[\left(\varepsilon_{d,t+1}^{(*1)} - \sqrt{\frac{h_{dt}^Q-\omega_d^Q}{2\omega_d^Q}}\right)^2 - \left(\frac{h_{dt}^Q+\omega_d^Q}{2\omega_d^Q}\right)\right] \\
&\quad + \left(\lambda_u-\xi_u\right)h_{ut} + \sqrt{\frac{\omega_u^Q}{2}}\left[\left(\frac{h_{ut}^Q+\omega_u^Q}{2\omega_u^Q}\right) - \kappa_1^u\left(\frac{h_{ut}+\omega_u}{2\omega_u}\right)\right] \\
&\quad + \sqrt{\frac{\omega_u^Q}{2}}\left[\left(\varepsilon_{u,t+1}^{(*1)} - \sqrt{\frac{h_{ut}^Q-\omega_u^Q}{2\omega_u^Q}}\right)^2 - \left(\frac{h_{ut}^Q+\omega_u^Q}{2\omega_u^Q}\right)\right].
\end{aligned}
$$

Invoking the no-arbitrage restriction implies that

$$
\begin{aligned}
\bar{r}^Q - \xi_d^Q h_{dt}^Q - \xi_u^Q h_{ut}^Q &= \bar{r} + \left(\lambda_d-\xi_d\right)h_{dt} - \sqrt{\frac{\omega_d^Q}{2}}\left[\left(\frac{h_{dt}^Q+\omega_d^Q}{2\omega_d^Q}\right) - \kappa_1^d\left(\frac{h_{dt}+\omega_d}{2\omega_d}\right)\right] \\
&\quad + \left(\lambda_u-\xi_u\right)h_{ut} + \sqrt{\frac{\omega_u^Q}{2}}\left[\left(\frac{h_{ut}^Q+\omega_u^Q}{2\omega_u^Q}\right) - \kappa_1^u\left(\frac{h_{ut}+\omega_u}{2\omega_u}\right)\right],
\end{aligned}
$$

where

$$\begin{aligned}
\bar{r}^Q &= r_f + \frac{1}{2}\left[\ln\left(1+\sqrt{2\omega_d^Q}\right) - \frac{\omega_d^Q + \sqrt{2\omega_d^Q}}{\left(1+\sqrt{2\omega_d^Q}\right)}\right] + \frac{1}{2}\left[\ln\left(1-\sqrt{2\omega_u^Q}\right) - \frac{\omega_u^Q - \sqrt{2\omega_u^Q}}{\left(1-\sqrt{2\omega_u^Q}\right)}\right], \\
\xi_d^Q &= \frac{1}{2\left(1+\sqrt{2\omega_d^Q}\right)},\ \xi_u^Q = \frac{1}{2\left(1-\sqrt{2\omega_u^Q}\right)}.
\end{aligned}$$

Therefore, the following identities hold

$$\begin{aligned}
\sqrt{\frac{\omega_d^Q}{2}}\left[\left(\frac{h_{dt}^Q + \omega_d^Q}{2\omega_d^Q}\right) - \kappa_1^d\left(\frac{h_{dt} + \omega_d}{2\omega_d}\right)\right] - \xi_d^Q h_{dt}^Q &= \bar{r}_d - \bar{r}_d^Q + (\lambda_d - \xi_d)\, h_{dt}, \\
-\sqrt{\frac{\omega_u^Q}{2}}\left[\left(\frac{h_{ut}^Q + \omega_u^Q}{2\omega_u^Q}\right) - \kappa_1^u\left(\frac{h_{ut} + \omega_u}{2\omega_u}\right)\right] - \xi_u^Q h_{ut}^Q &= \bar{r}_u - \bar{r}_u^Q + (\lambda_u - \xi_u)\, h_{ut}.
\end{aligned}$$

Namely, we have

$$h_{jt}^Q \equiv \vartheta_j + \varsigma_j h_{jt},\ \text{for } j = u, d,$$

with

$$\begin{aligned}
\varsigma_d &= \frac{\sqrt{2\omega_d^Q}\left(1+\sqrt{2\omega_d^Q}\right)}{\sqrt{2\omega_d}\left(1+\sqrt{2\omega_d}\right)} + 2\sqrt{2\omega_d^Q}\left(1+\sqrt{2\omega_d^Q}\right)\lambda_d, \\
\varsigma_u &= \frac{\sqrt{2\omega_u^Q}\left(1-\sqrt{2\omega_u^Q}\right)}{\sqrt{2\omega_u}\left(1-\sqrt{2\omega_u}\right)} - 2\sqrt{2\omega_u^Q}\left(1-\sqrt{2\omega_u^Q}\right)\lambda_u,
\end{aligned}$$

and

$$\begin{aligned}
\vartheta_d &= \sqrt{2\omega_d^Q}\left(1+\sqrt{2\omega_d^Q}\right)\ln\left(\frac{1+\sqrt{2\omega_d}}{1+\sqrt{2\omega_d^Q}}\right) + \frac{\sqrt{\omega_d^Q}\left(\sqrt{\omega_d^Q} - \sqrt{\omega_d}\right)}{1+\sqrt{2\omega_d}}, \\
\vartheta_u &= \sqrt{2\omega_u^Q}\left(1-\sqrt{2\omega_u^Q}\right)\ln\left(\frac{1-\sqrt{2\omega_u^Q}}{1-\sqrt{2\omega_u}}\right) + \frac{\sqrt{\omega_u^Q}\left(\sqrt{\omega_u^Q} - \sqrt{\omega_u}\right)}{1-\sqrt{2\omega_u}}.
\end{aligned}$$

The variance spread is defined as

$$\begin{aligned} h_{jt}^{Q} - h_{jt} &= \vartheta_j + \varsigma_j h_{jt} - h_{jt} \\ &= \vartheta_j + \left(\varsigma_j - 1\right) h_{jt}, \end{aligned}$$

where

$$\begin{aligned} \varsigma_d - 1 &= \frac{\sqrt{2\omega_d^{Q}}\left(1+\sqrt{2\omega_d^{Q}}\right)}{\sqrt{2\omega_d}\left(1+\sqrt{2\omega_d}\right)} + 2\sqrt{2\omega_d^{Q}}\left(1+\sqrt{2\omega_d^{Q}}\right)\lambda_d - 1 \\ &= \frac{\sqrt{2\omega_d^{Q}}\left(1+\sqrt{2\omega_d^{Q}}\right)}{\sqrt{2\omega_d}\left(1+\sqrt{2\omega_d}\right)} - 1 + 2\sqrt{2\omega_d^{Q}}\left(1+\sqrt{2\omega_d^{Q}}\right)\lambda_d, \end{aligned}$$

$$h_{jt+1}^{Q} - \omega_j^{Q} = \varpi_j^{Q} + \beta_j\left(h_{jt} - \omega_j^{Q}\right) + \alpha_j^{Q}\left(\varepsilon_{jt+1}^{*(2)} - \gamma_j^{Q}\sqrt{h_{jt}^{Q} - \omega_j^{Q}}\right)^2,$$

$$\varpi_j^{Q} = \left(1-\beta_j\right)\left(\vartheta_j + \varsigma_j\omega_j - \omega_j^{Q}\right) + \varsigma_j\varpi_j,$$

$$\alpha_j^{Q} = \frac{\varsigma_j\alpha_j}{\kappa_2^{j}},$$

$$\rho_j^{Q} = \frac{\rho_j}{\sqrt{1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}}\sqrt{1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}}}.$$

We use the following equality

$$\begin{aligned} &\left(1-2\left(1-\rho_j^2\right)\nu_2^{(1j)}\right)\left(1-2\left(1-\rho_j^2\right)\nu_2^{(2j)}\right) \\ = \ &\left(1-\rho_j^2\right)\frac{\left(1-\rho_j^2\right)\kappa_1^{j}\kappa_2^{j} + \sqrt{\left(1-\rho_j^2\right)^2\left(\kappa_1^{j}\right)^2\left(\kappa_2^{j}\right)^2 + 4\rho_j^2\kappa_1^{j}\kappa_2^{j}}}{2} + \rho_j^2, \end{aligned}$$

to prove that the dynamics of the $j-$side realized semi-variance under the risk-neutral distribution can be cast as

$$
\begin{aligned}
RV_{t+1}^{j} &= h_{jt}+\sigma_j\left[\left(\varepsilon_{jt+1}^{(2)}-\gamma_j\sqrt{h_{jt}-\omega_j}\right)^2-\left(1+\gamma_j^2\left(h_{jt}-\omega_j\right)\right)\right],\\
&= h_{jt}+\sigma_j\left[\frac{1}{\kappa_2^j}\left(\varepsilon_{jt+1}^{*(2)}-\gamma_j^Q\sqrt{h_{jt}^Q-\omega_j^Q}\right)^2-\left(1+\gamma_j^2\left(h_{jt}-\omega_j\right)\right)\right],\\
&= h_{jt}+\frac{\sigma_j}{\kappa_2^j}\left[\left(\varepsilon_{jt+1}^{*(2)}-\gamma_j^Q\sqrt{h_{jt}^Q-\omega_j^Q}\right)^2-\kappa_2^j\left(1+\gamma_j^2\left(h_{jt}-\omega_j\right)\right)\right].
\end{aligned}
$$

Next, we derive the expression of the variance risk premium

$$
\begin{aligned}
& E_t^Q\left[RV_{t+1}^j\right]-E_t^P\left[RV_{t+1}^j\right]\\
= & \frac{\sigma_j}{\kappa_2^j}\left[\left(1+\left(\gamma_j^Q\right)^2\left(h_{jt}^Q-\omega_j^Q\right)\right)-\kappa_2^j\left(1+\gamma_j^2\left(h_{jt}-\omega_j\right)\right)\right],\\
= & \frac{\sigma_j}{\kappa_2^j}\left[1-\left(\gamma_j^Q\right)^2\omega_j^Q-\kappa_2^j+\kappa_2^j\gamma_j^2\omega_j+\left(\gamma_j^Q\right)^2h_{jt}^Q-\kappa_2^j\gamma_j^2h_{jt}\right],\\
= & \frac{\sigma_j}{\kappa_2^j}\left[1-\left(\gamma_j^Q\right)^2\omega_j^Q-\kappa_2^j+\kappa_2^j\gamma_j^2\omega_j+\left(\gamma_j^Q\right)^2\left(h_{jt}^Q-h_{jt}+h_{jt}\right)-\kappa_2^j\gamma_j^2h_{jt}\right],\\
= & \frac{\sigma_j}{\kappa_2^j}\left[1-\left(\gamma_j^Q\right)^2\omega_j^Q-\kappa_2^j+\kappa_2^j\gamma_j^2\omega_j+\left(\gamma_j^Q\right)^2\left(h_{jt}^Q-h_{jt}\right)+\left(\left(\gamma_j^Q\right)^2-\kappa_2^j\gamma_j^2\right)h_{jt}\right].
\end{aligned}
$$

The parameters of the pricing kernel are

$$
\begin{aligned}
\nu_{1t}^{(1j)} &= \pi_{1j}\sqrt{h_{j,t}-\omega_j}+\pi_{1j}^Q\sqrt{h_{jt}^Q-\omega_j^Q},\\
\nu_{1t}^{(2j)} &= \pi_{2j}\sqrt{h_{j,t}-\omega_j}+\pi_{2j}^Q\sqrt{h_{jt}^Q-\omega_j^Q},\\
\nu_{2}^{(1j)} &= \frac{1}{2\left(1-\rho_j^2\right)}\left(1-\frac{\sigma_j^Q}{\sigma_j\delta_j^2}\right)+\frac{1}{2}\left(\frac{\sigma_j^Q}{\sigma_j\delta_j^2}-\sqrt{\frac{\omega_j}{\omega_j^Q}}\right),\\
\nu_{2}^{(2j)} &= \frac{1}{2\left(1-\rho_j^2\right)}\left(1-\frac{\sigma_j}{\sigma_j^Q}\ \delta_j^2\right),
\end{aligned}
$$

with

$$
\begin{aligned}
\pi_{1j} &= \frac{1}{1-\left(\rho_j^Q\right)^2}\left(\frac{1}{\sqrt{2\omega_j^Q}} - \gamma_j \frac{\rho_j}{\delta_j^2}\right),\ \pi_{1j}^Q = \frac{1}{1-\left(\rho_j^Q\right)^2}\left(\frac{1}{\sqrt{2\omega_j^Q}}\left(\frac{\omega_j}{\omega_j^Q}\right)^{1/4} - \gamma_j^Q \frac{\rho_j}{\delta_j^2}\left(\frac{\sigma_j^Q}{\sigma_j}\right)^{1/2}\right), \\
\pi_{2j} &= \frac{1}{1-\left(\rho_j^Q\right)^2}\left(\gamma_j \frac{\sigma_j}{\sigma_j^Q} - \frac{\sigma_j \delta_j^2 \rho_j}{\left(1-\rho_j^2\right)\delta_j^2 \sigma_j \sqrt{2\omega_j} + \rho_j^2 \sigma_j^Q \sqrt{2\omega_j^Q}}\right), \\
\pi_{2j}^Q &= \frac{1}{1-\left(\rho_j^Q\right)^2}\left(\gamma_j^Q \left(\frac{\sigma_j}{\sigma_j^Q}\right)^{1/2} - \frac{\sigma_j \delta_j^2 \rho_j}{\left(1-\rho_j^2\right)\delta_j^2 \sigma_j \sqrt{2\omega_j} + \rho_j^2 \sigma_j^Q \sqrt{2\omega_j^Q}}\left(\frac{\omega_j}{\omega_j^Q}\right)^{1/4}\right).
\end{aligned}
$$

Finally, we apply the general risk neutralization procedure to each of the models, and provide the corresponding risk-neutral dynamics. Specific formulas for one-factor models are presented below.

C.1 Heston and Nandi Affine GARCH model

$$
\begin{aligned}
R_{t+1} &= r - \frac{1}{2}h_t^Q + \varepsilon_{t+1}^*, \\
\varepsilon_{t+1} &= \sqrt{h_t^Q} z_{t+1}^*, \\
h_{t+1}^Q &= \omega^Q + \beta h_t^Q + \alpha^Q \left(z_{t+1}^* - \gamma^Q \sqrt{h_t^Q}\right)^2,
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma^Q &= \gamma^* \sqrt{\frac{\alpha}{\alpha^*}} = \frac{\alpha}{\alpha^*}\left(\lambda + \gamma - \frac{1}{2}\right) + \frac{1}{2} \\
&= \kappa\left(\lambda + \gamma - \frac{1}{2}\right) + \frac{1}{2}, \\
\alpha^Q &= \frac{(\alpha^*)^2}{\alpha} = \alpha \frac{(\alpha^*)^2}{\alpha^2} = \frac{\alpha}{\kappa^2}, \\
\omega^Q &= \frac{\alpha^*}{\alpha}\omega = \frac{\omega}{\kappa}, \\
h_t^Q &= \frac{h_t}{\kappa}.
\end{aligned}
$$

C.2 ARV

$$\begin{aligned}
R_{t+1} &= r - \frac{h_t^Q}{2} + \sqrt{h_t^Q}\varepsilon_{t+1}^{*(1)}, \\
h_t^Q &= \frac{h_t}{\kappa_1}, \\
h_{t+1}^Q &= \omega^Q + \beta h_t^Q + \alpha^Q \left(\varepsilon_{t+1}^{*(2)} - \gamma^Q \sqrt{h_t^Q}\right)^2, \\
\omega^Q &= \frac{\omega}{\kappa_1}, \quad \alpha^Q = \frac{\alpha}{\kappa_1 \kappa_2}, \\
E^Q\left[\varepsilon_{t+1}^{*(1)} \varepsilon_{t+1}^{*(2)}\right] &= \rho^Q,
\end{aligned}$$

with

$$\rho^Q = \frac{\rho}{\sqrt{(1-\rho^2)\frac{(1-\rho^2)\kappa_1\kappa_2 + \sqrt{(1-\rho^2)^2\kappa_1^2\kappa_2^2 + 4\rho^2\kappa_1\kappa_2}}{2} + \rho^2}},$$

$$\begin{aligned}
RV_{t+1} &= h_t + \sigma^Q \left[1 - \kappa_2 + \left(\frac{\left(\gamma^Q\right)^2}{\kappa_1} - \gamma^2 \kappa_2\right) h_t\right] \\
&\quad + \sigma^Q \left[\left(\varepsilon_{t+1}^{*(2)} - \gamma^Q \sqrt{h_t^Q}\right)^2 - \left(1 + \left(\gamma^Q\right)^2 h_t^Q\right)\right],
\end{aligned}$$

with

$$\sigma^Q = \frac{\sigma}{\kappa_2}.$$

C.3 SARV

$$R_{t+1} = \bar{r}^Q - \xi^Q h_t^Q - \sqrt{\frac{\omega^Q}{2}} \left[\left(\varepsilon_{t+1}^{*(1)} - \sqrt{\frac{h_t^Q - \omega^Q}{2\omega^Q}}\right)^2 - \left(\frac{h_t^Q + \omega^Q}{2\omega^Q}\right)\right],$$

where

$$\begin{aligned}
\bar{r}^Q &= r_f + \frac{1}{2}\left[\ln\left(1 + \sqrt{2\omega^Q}\right) - \frac{\omega^Q + \sqrt{2\omega^Q}}{\left(1 + \sqrt{2\omega^Q}\right)}\right], \\
\xi^Q &= \frac{1}{2\left(1 + \sqrt{2\omega^Q}\right)},
\end{aligned}$$

The mapping between the risk-neutral and the physical variances is given by

$$\begin{aligned} h_t^Q &\equiv \vartheta + \varsigma h_t, \\ \vartheta &= \sqrt{2\omega^Q}\left(1+\sqrt{2\omega^Q}\right)\ln\left(\frac{1+\sqrt{2\omega}}{1+\sqrt{2\omega^Q}}\right) + \frac{\sqrt{\omega^Q}\left(\sqrt{\omega^Q}-\sqrt{\omega}\right)}{1+\sqrt{2\omega}}, \\ \varsigma &= 2\sqrt{2\omega^Q}\left(1+\sqrt{2\omega^Q}\right)\lambda + \frac{\sqrt{2\omega^Q}\left(1+\sqrt{2\omega^Q}\right)}{\sqrt{2\omega}\left(1+\sqrt{2\omega}\right)}. \end{aligned}$$

The realized variance dynamics under the risk-neutral measure is

$$\begin{aligned} RV_{t+1} &= h_t + \sigma^Q\left[\left(1+\left(\gamma^Q\right)^2\left(h_t^Q-\omega^Q\right)\right) - \kappa_2\left(1+\gamma^2\left(h_t-\omega\right)\right)\right], \\ &\quad +\sigma^Q\left[\left(\varepsilon_{t+1}^{*(2)} - \gamma^Q\sqrt{h_t^Q-\omega^Q}\right)^2 - \left(1+\left(\gamma^Q\right)^2\left(h_t^Q-\omega^Q\right)\right)\right], \end{aligned}$$

where

$$h_{t+1}^Q - \omega^Q = \varpi^Q + \beta\left(h_t^Q-\omega^Q\right) + \alpha^Q\left(z_{t+1}^{*(2)} - \gamma^Q\sqrt{h_t^Q-\omega^Q}\right)^2,$$

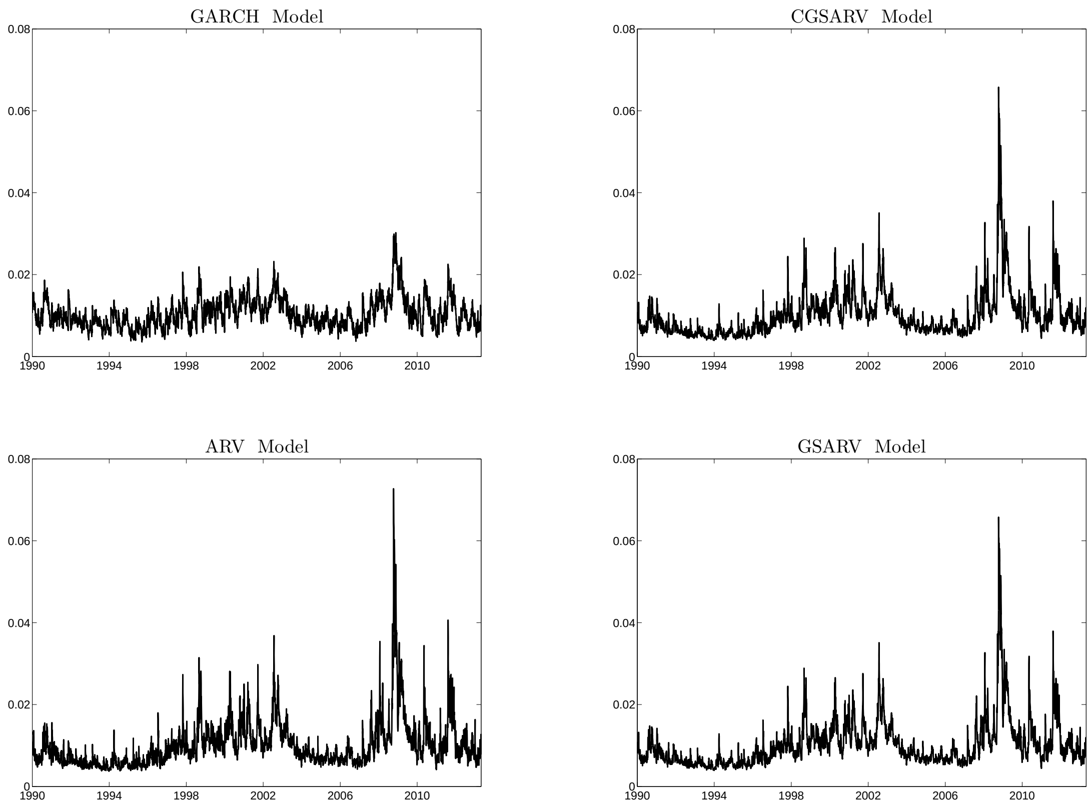
$$\begin{aligned} \varpi^Q &= (1-\beta)\left(\vartheta+\varsigma\omega-\omega^Q\right)+\varsigma\varpi, \\ \alpha^Q &= \frac{\varsigma\sigma^Q\alpha}{\sigma}, \\ \rho^Q &\equiv E^Q\left[\varepsilon_{t+1}^{*(1)}\varepsilon_{t+1}^{*(2)}\right] = \frac{\rho}{\sqrt{\left(1-\rho^2\right)\frac{(1-\rho^2)\kappa_1\kappa_2+\sqrt{(1-\rho^2)^2\kappa_1^2\kappa_2^2+4\rho^2\kappa_1\kappa_2}}{2}+\rho^2}}, \end{aligned}$$

$$\kappa_1 = \sqrt{\frac{\omega}{\omega^Q}}, \text{ and } \kappa_2 = \frac{\sigma}{\sigma^Q}.$$

C.4 GSARV

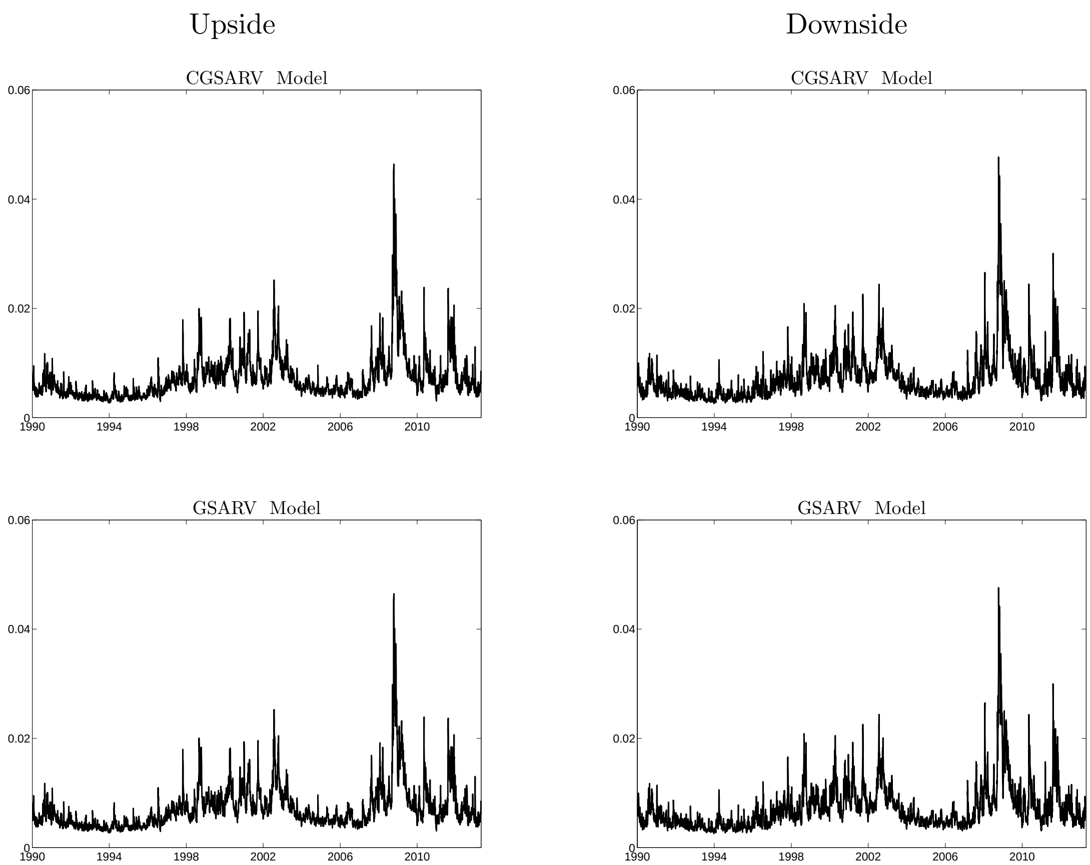
The computations follow the same steps as above given that the GSARV is a straightforward two-factor extension of the single factor SARV.

Figure A1: Daily Conditional Volatilities



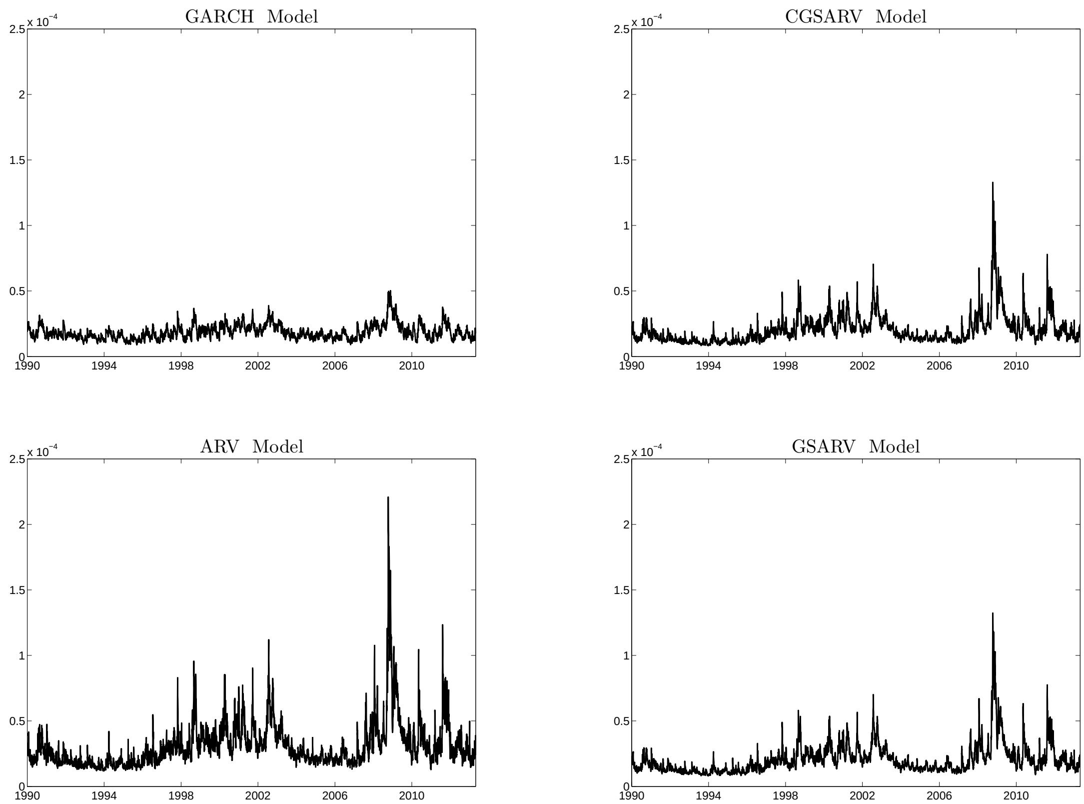
These figures present the daily conditional volatilities, $\sqrt{h_t}$, implied by the parameters estimated for each model in the historical optimization from January 02, 1990 through August 28, 2013.

Figure A2: Daily Upside and Downside Conditional Volatilities



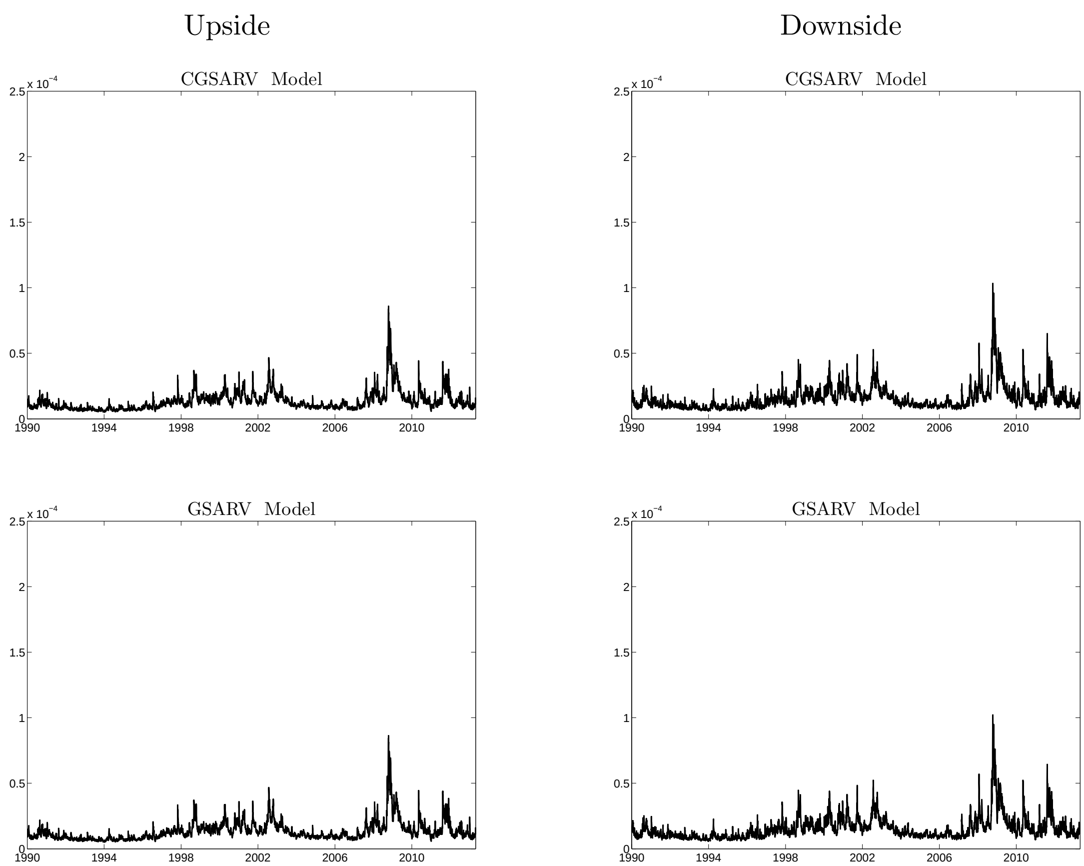
These figures present daily upside (left column) and downside (right column) conditional volatilities, $\sqrt{h_{u,t}}$ and $\sqrt{h_{d,t}}$, implied by the parameters estimated for two-factor models in the historical optimization from January 02, 1990 through August 28, 2013.

Figure A3: Daily Conditional Volatilities of Variances



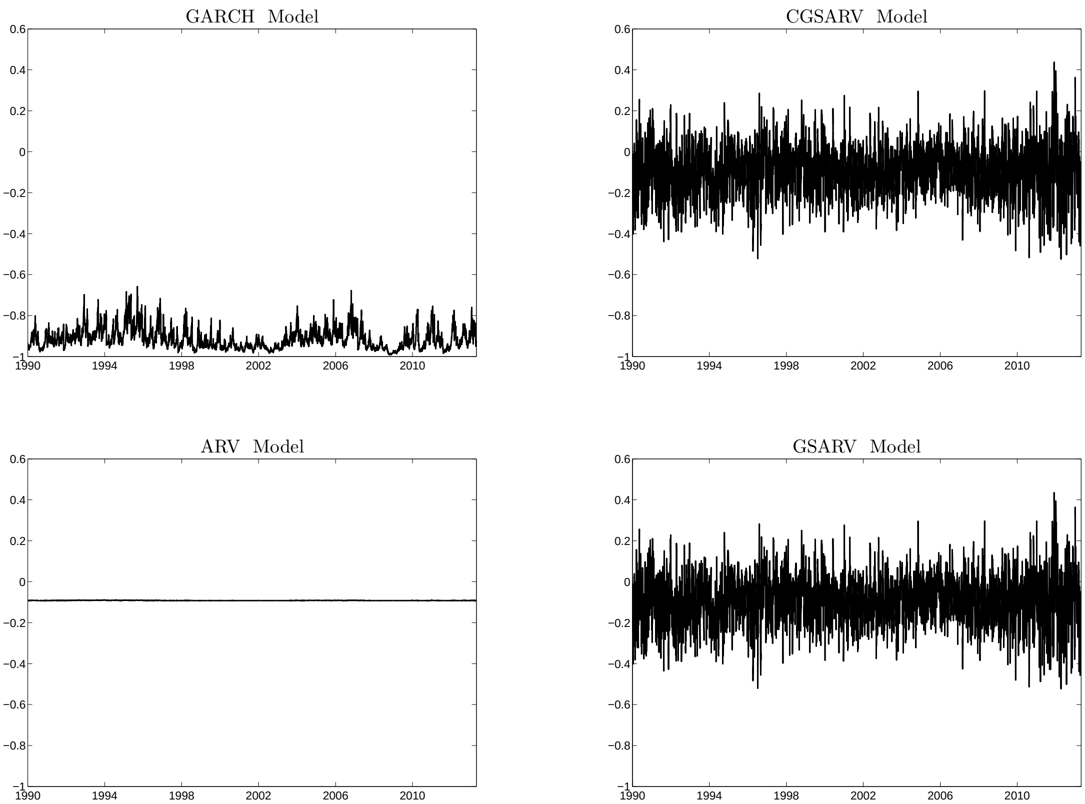
These figures present the daily conditional volatilities of variances, $\sqrt{Var_t(h_{t+1})}$, implied by the parameters estimated for each model in the historical optimization from January 02, 1990 through August 28, 2013.

Figure A4: Daily Conditional Volatilities of Upside and Downside Variances



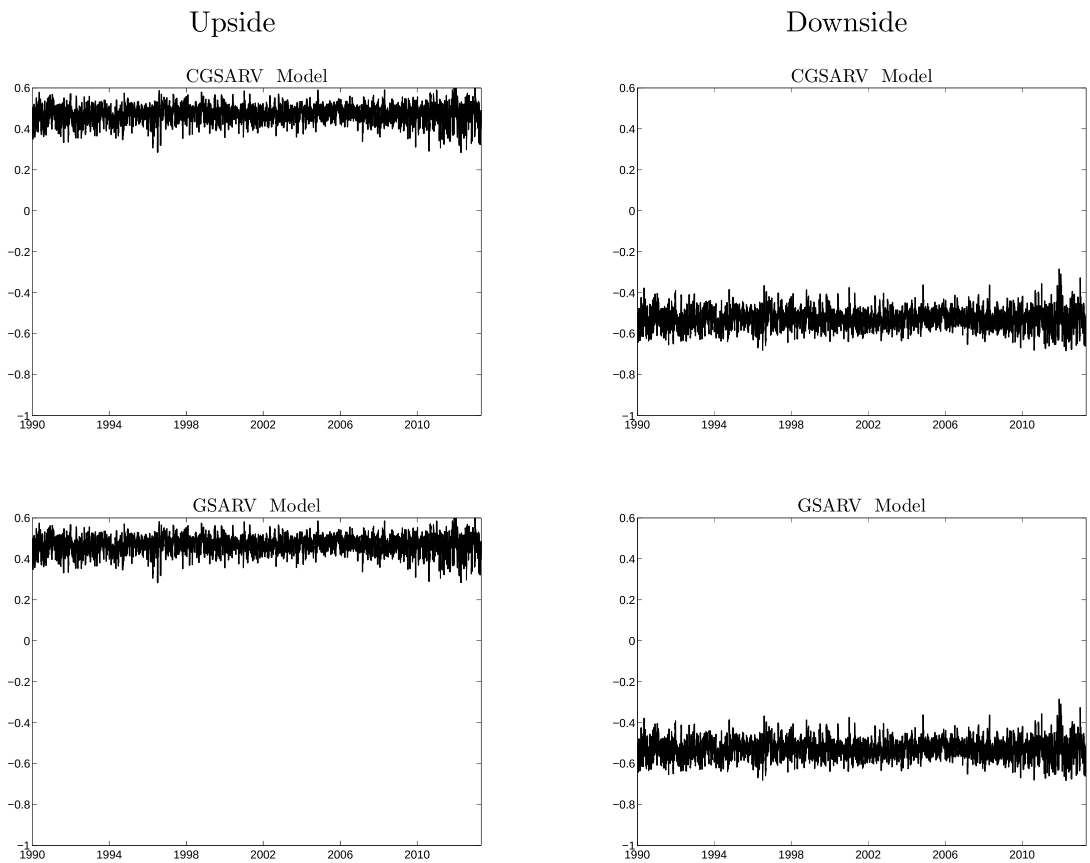
These figures present daily conditional volatilities of upside (left column) and downside (right column) variances, $\sqrt{Var_t(h_{u,t+1})}$ and $\sqrt{Var_t(h_{d,t+1})}$, implied by the parameters estimated for two-factor models in the historical optimization from January 02, 1990 through August 28, 2013.

Figure A5: Daily Conditional Correlations between Returns and Variances



These figures present the daily conditional correlations between returns and variances, $Corr_t(R_{t+1}, h_{t+1})$, implied by the parameters estimated for each model in the historical optimization from January 02, 1990 through August 28, 2013.

Figure A6: Daily Conditional Correlations between Returns and Upside, and Downside Variances



These figures present daily conditional correlations between returns and upside (left column), and downside (right column) variances, $Corr_t(R_{t+1}, h_{u,t+1})$ and $Corr_t(R_{t+1}, h_{d,t+1})$, implied by the parameters estimated for two-factor models in the historical optimization from January 02, 1990 through August 28, 2013.