A Appendix

A Monetary Policy

A.1 Joint Conditional Moment Generating Function

Lemma 1

$$\begin{aligned} E\left[\exp\left(u'\eta_{t+1}+\eta_{t+1}'\Gamma\eta_{t+1}\right)\right] &= \exp\left(-\frac{1}{2}\ln\det\left(I-2\Sigma\Sigma'\Gamma\right)+\frac{1}{2}u'\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}u\right) \\ E_t\left[\exp\left(u'Y_{t+1}+Y_{t+1}'\Gamma Y_{t+1}\right)\right] &= \exp\left(A\left(u,\Gamma\right)+B\left(u,\Gamma\right)'Y_t+Y_t'C\left(\Gamma\right)Y_t\right) \end{aligned}$$

with

$$\begin{aligned} A\left(u,\Gamma\right) &= u'\omega+\omega'\Gamma\omega-\frac{1}{2}\ln\det\left(I-2\Sigma\Sigma'\Gamma\right) \\ &+\frac{1}{2}\left(u+\Gamma'\omega+\Gamma\omega\right)'\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}\left(u+\Gamma'\omega+\Gamma\omega\right) \\ B\left(u,\Gamma\right) &= \phi'\left\{\left[\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}+\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma'\right)^{-1}\right]\frac{\Gamma'+\Gamma}{2}+I\right\}\left(u+\Gamma'\omega+\Gamma\omega\right) \\ C\left(\Gamma\right) &= \frac{1}{2}\phi'\left[\left(\Gamma'+\Gamma\right)\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}\left(\Gamma'+\Gamma\right)+2\Gamma\right]\phi \end{aligned}$$

$$E_t\left[\exp\left(u'Y_{t+h}+Y_{t+h}'\Gamma Y_{t+h}\right)\right]=\exp\left(A\left(u,\Gamma;h\right)+B\left(u,\Gamma;h\right)'Y_t+Y_t'C\left(\Gamma;h\right)Y_t\right)$$

where

$$\begin{aligned} A\left(u,\Gamma;h+1\right) &= A\left(u,\Gamma;h\right)+A\left(B\left(u,\Gamma;h\right),C\left(\Gamma;h\right)\right) \\ B\left(u,\Gamma;h+1\right) &= B\left(B\left(u,\Gamma;h\right),C\left(\Gamma;h\right)\right) \\ C\left(\Gamma;h+1\right) &= C\left(C\left(\Gamma;h\right)\right) \end{aligned}$$

with the following initial conditions

$$A\left(u,\Gamma;0\right)=0,\quad ,B\left(u,\Gamma;0\right)=u,\quad ,C\left(\Gamma;0\right)=\Gamma$$

$$\begin{aligned} & u'Y_{t+1}+Y_{t+1}'\Gamma Y_{t+1} \\ = \; & u'\left(\omega+\phi Y_t+\eta_{t+1}\right)+\left(\omega+\phi Y_t+\eta_{t+1}\right)'\Gamma\left(\omega+\phi Y_t+\eta_{t+1}\right) \\ = \; & u'\left(\omega+\phi Y_t+\eta_{t+1}\right)+\omega'\Gamma\omega+\omega'\Gamma\phi Y_t+\omega'\Gamma\eta_{t+1}+Y_t'\phi'\Gamma\omega \\ & +Y_t'\phi'\Gamma\phi Y_t+Y_t'\phi'\Gamma\eta_{t+1}+\eta_{t+1}'\Gamma\omega+\eta_{t+1}'\Gamma\phi Y_t+\eta_{t+1}'\Gamma\eta_{t+1} \\ = \; & u'\omega+\omega'\Gamma\omega+\left(u'\phi+\omega'\Gamma\phi+\omega'\Gamma'\phi\right)Y_t+Y_t'\phi'\Gamma\phi Y_t \\ & +\left(u'+\omega'\Gamma+\omega'\Gamma'+Y_t'\phi'\Gamma+Y_t'\phi'\Gamma'\right)\eta_{t+1}+\eta_{t+1}'\Gamma\eta_{t+1} \end{aligned}$$

$$
\begin{aligned}
& E_t\left[\exp\left(u'Y_{t+1}+Y'_{t+1}\Gamma Y_{t+1}\right)\right] \\
= \quad & \exp\left(\begin{array}{c} u'\omega+\omega'\Gamma\omega-\frac{1}{2}\ln\det\left(I-2\Sigma\Sigma'\Gamma\right)+\left(u+\Gamma'\omega+\Gamma\omega\right)'\phi Y_t+Y_t'\phi'\Gamma\phi Y_t \\ +\frac{1}{2}\left(u+\Gamma'\omega+\Gamma\omega+\Gamma'\phi Y_t+\Gamma\phi Y_t\right)'\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}\left(u+\Gamma'\omega+\Gamma\omega+\Gamma'\phi Y_t+\Gamma\phi Y_t\right) \end{array}\right) \\
= \quad & \exp\left(\begin{array}{c} u'\omega+\omega'\Gamma\omega-\frac{1}{2}\ln\det\left(I-2\Sigma\Sigma'\Gamma\right) \\ +\frac{1}{2}\left(u+\Gamma'\omega+\Gamma\omega\right)'\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}\left(u+\Gamma'\omega+\Gamma\omega\right) \\ +\left(u+\Gamma'\omega+\Gamma\omega\right)'\left\{\left[\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}+\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma'\right)^{-1}\right]\frac{\Gamma'+\Gamma}{2}+I\right\}\phi Y_t \\ +\frac{1}{2}Y_t'\phi'\left[\left(\Gamma'+\Gamma\right)'\left(\left(\Sigma\Sigma'\right)^{-1}-2\Gamma\right)^{-1}\left(\Gamma'+\Gamma\right)+2\Gamma\right]\phi Y_t \end{array}\right)
\end{aligned}
$$

$$
X_{t+1}=\left(r_{t+1},Y'_{t+1}\right)'
$$

$$
\begin{aligned}
E_t\left[\exp\left(u'X_{t+1}\right)\right] &= E_t\left[\exp\left(u_r r_{t+1}+u_y'Y_{t+1}\right)\right]=E_t\left[\exp\left(u_y'Y_{t+1}\right)E_t\left[\exp\left(u_r r_{t+1}\right)|Y_{t+1}\right]\right] \\
&= E_t\left[\exp\left(u_y'Y_{t+1}\right)E_t\left[\exp\left(u_r\left(J_{t+1}r_t+c\bar{\Delta}_{t+1}\right)\right)|Y_{t+1}\right]\right] \\
&= E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t\right)E_t\left[\exp\left(u_rc\bar{\Delta}_{t+1}\right)|Y_{t+1}\right]\right] \\
&= \sum_{n=0}^{\bar{n}}E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn\right)P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
E_t\left[\exp\left(u'X_{t+1}\right)\right] &= \sum_{n=0}^{\bar{n}}E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn\right)CDF_{t|Y_{t+1}}\left(\delta_{n,t}\left(Y_{t+1}\right)\right)\right] \\
&\quad -\sum_{n=0}^{\bar{n}}E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn\right)CDF_{t|Y_{t+1}}\left(\delta_{n-1,t}\left(Y_{t+1}\right)\right)\right]
\end{aligned}
$$

Given that these conditions are satisfied, we have

$$
\begin{aligned}
& E_t\left[\exp\left(u'X_{t+1}\right)\right] \\
= \quad & \sum_{n=0}^{\bar{n}}E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn+\alpha_n+\beta_n'Y_{t+1}+Y'_{t+1}\Gamma_nY_{t+1}\right)\right] \\
& -\sum_{n=0}^{\bar{n}}E_t\left[\exp\left(u_y'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn+\alpha_{n-1}+\beta_{n-1}'Y_{t+1}+Y'_{t+1}\Gamma_{n-1}Y_{t+1}\right)\right] \\
= \quad & \sum_{n=0}^{\bar{n}}E_t\left[\exp\left(\left(u_y+\beta_n\right)'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn+\alpha_n+Y'_{t+1}\Gamma_nY_{t+1}\right)\right] \\
& -\sum_{n=0}^{\bar{n}}E_t\left[\exp\left(\left(u_y+\beta_{n-1}\right)'Y_{t+1}+u_rJ_{t+1}r_t+u_rcn+\alpha_{n-1}+Y'_{t+1}\Gamma_{n-1}Y_{t+1}\right)\right] \\
= \quad & \sum_{n=0}^{\bar{n}}\exp\left(\begin{array}{c} u_rJ_{t+1}r_t+u_rcn+\alpha_n+A\left(u_y+\beta_n,\Gamma_n\right) \\ +B\left(u_y+\beta_n,\Gamma_n\right)'Y_t+Y_t'C\left(\Gamma_n\right)Y_t \end{array}\right) \\
& -\sum_{n=0}^{\bar{n}}\exp\left(\begin{array}{c} u_rJ_{t+1}r_t+u_rcn+\alpha_{n-1}+A\left(u_y+\beta_{n-1},\Gamma_{n-1}\right) \\ +B\left(u_y+\beta_{n-1},\Gamma_{n-1}\right)'Y_t+Y_t'C\left(\Gamma_{n-1}\right)Y_t \end{array}\right)
\end{aligned}
$$

Let assume that $J_{t+1}=0, \quad \forall t$ we have

$$
\begin{aligned}
& E_t\left[\exp\left(u' X_{t+1}\right)\right] \\
= & \sum_{n=0}^{\bar{n}} \exp\left(\begin{array}{c} u_r c n+\alpha_n+A\left(u_y+\beta_n, \Gamma_n\right) \\ +B\left(u_y+\beta_n, \Gamma_n\right)' Y_t+Y_t' C\left(\Gamma_n\right) Y_t \end{array}\right) \\
& -\exp\left(u_r c\right) \sum_{m=-1}^{\bar{n}-1} \exp\left(\begin{array}{c} u_r c m+\alpha_m+A\left(u_y+\beta_m, \Gamma_m\right) \\ +B\left(u_y+\beta_m, \Gamma_m\right)' Y_t+Y_t' C\left(\Gamma_m\right) Y_t \end{array}\right) \\
= & \exp\left(u_r c \bar{n}+A\left(u_y, 0\right)+B\left(u_y, 0\right)' Y_t+Y_t' C(0) Y_t\right)+ \\
& \left(1-\exp\left(u_r c\right)\right) \sum_{n=0}^{\bar{n}-1} \exp\left(\begin{array}{c} u_r c n+\alpha_n+A\left(u_y+\beta_n, \Gamma_n\right) \\ +B\left(u_y+\beta_n, \Gamma_n\right)' Y_t+Y_t' C\left(\Gamma_n\right) Y_t \end{array}\right) \\
= & \sum_{n=0}^{\bar{n}} \exp\left(A_n(u)+B_n(u)' Y_t+Y_t' C_n(u) Y_t\right)
\end{aligned}
$$

with

$$
\begin{aligned}
A_n(u) & = \ln\left(1-\exp\left(u_r c\right)\right)+u_r c n+\alpha_n+A\left(u_y+\beta_n, \Gamma_n\right) \\
B_n(u) & = B\left(u_y+\beta_n, \Gamma_n\right), \quad , C_n(u)=C\left(\Gamma_n\right) \\
for\ n & < \bar{n}
\end{aligned}
$$

and

$$
A_{\bar{n}}(u)=u_r c \bar{n}+A\left(u_y, 0\right), \quad , B_{\bar{n}}(u)=B\left(u_y, 0\right), \quad , C_{\bar{n}}(u)=0
$$

A.2 Conditional Cumulative Probability Function

$$
\begin{aligned}
& P_{t \mid Y_{t+1}}\left[\bar{\Delta}_{t+h} \leq n\right] \\
= & E_{t \mid Y_{t+1}}\left[1_{\left[\bar{\Delta}_{t+h} \leq n\right]}\right]=E_{t \mid Y_{t+1}}\left[E_{t+h-1 \mid Y_{t+h}}\left[1_{\left[\bar{\Delta}_{t+h} \leq n\right]}\right]\right] \\
= & E_{t \mid Y_{t+1}}\left[\exp\left(\alpha_n+\beta_n' Y_{t+h}+Y_{t+h}' \Gamma_n Y_{t+h}\right)\right] \\
= & \exp\left(\alpha_n+A\left(\beta_n, \Gamma_n ; h-1\right)+B\left(\beta_n, \Gamma_n ; h-1\right)' Y_{t+1}+Y_{t+1}' C\left(\Gamma_n ; h-1\right) Y_{t+1}\right)
\end{aligned}
$$

$$
P_t\left[\bar{\Delta}_{t+h} \leq n\right]=\exp\left(\alpha_n+A\left(\beta_n, \Gamma_n ; h\right)+B\left(\beta_n, \Gamma_n ; h\right)' Y_t+Y_t' C\left(\Gamma_n ; h\right) Y_t\right)
$$

A.3 Temporal Aggregation

$$\begin{aligned}
&\varphi_{t,h}\left(u_1,...,u_h\right) \\
\equiv\ & E_t\left[\exp\left(\sum_{j=1}^{h} u_j' X_{t+j}\right)\right] \\
=\ & E_t\left[\sum_{n=0}^{\bar{n}} \exp\left(A_n(u_h) + B_n\left(u_h\right)' Y_{t+h-1} + Y_{t+h-1}' C_n\left(u_h\right) Y_{t+h-1} + \sum_{j=1}^{h-1} u_j' X_{t+j}\right)\right] \\
=\ & E_t\left[\sum_{n=0}^{\bar{n}} \exp\left(\begin{array}{c} A_n(u_h) + \sum_{j=1}^{h-2} u_j' X_{t+j} + \left(B_n\left(u_h\right) + u_{y,h-1}\right)' Y_{t+h-1} \\ +Y_{t+h-1}' C_n\left(u_h\right) Y_{t+h-1} + u_{r,h-1} r_{t+h-1} \end{array}\right)\right] \\
=\ & E_t\left[\sum_{n=0}^{\bar{n}} \sum_{n_{h-1}=0}^{\bar{n}} \exp\left(\begin{array}{c} A_n(u_h) + \sum_{j=1}^{h-2} u_j' X_{t+j} + \left(B_n\left(u_h\right) + u_{y,h-1}\right)' Y_{t+h-1} \\ +Y_{t+h-1}' C_n\left(u_h\right) Y_{t+h-1} + \bar{\gamma}_{0,n}\left(u_{r,h-1}\right) + \beta_n' Y_{t+h-1} + Y_{t+h-1}' \Gamma_n Y_{t+h-1} \end{array}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\varphi_{t,h}\left(u_1,...,u_h\right) \\
=\ & E_t\left[\sum_{n=0}^{\bar{n}} \sum_{n_{h-1}=0}^{\bar{n}} \exp\left(\begin{array}{c} A_n(u_h) + \bar{\gamma}_{0,n_{h-1}}\left(u_{r,h-1}\right) + \sum_{j=1}^{h-2} u_j' X_{t+j} \\ + \left(B_n\left(u_h\right) + u_{y,h-1} + \beta_{n_{h-1}}\right)' Y_{t+h-1} \\ +Y_{t+h-1}' \left(C_n\left(u_h\right) + \Gamma_{n_{h-1}}\right) Y_{t+h-1} \end{array}\right)\right] \\
=\ & E_t\left[\sum_{n=0}^{\bar{n}} \sum_{n_{h-1}=0}^{\bar{n}} \exp\left(\begin{array}{c} A_n(u_h) + \bar{\gamma}_{0,n_{h-1}}\left(u_{r,h-1}\right) + \sum_{j=1}^{h-2} u_j' X_{t+j} \\ +A\left(B_n\left(u_h\right) + u_{y,h-1} + \beta_{n_{h-1}}, C_n\left(u_h\right) + \Gamma_{n_{h-1}}\right) \\ +B\left(B_n\left(u_h\right) + u_{y,h-1} + \beta_{n_{h-1}}, C_n\left(u_h\right) + \Gamma_{n_{h-1}}\right)' Y_{t+h-2} \\ +Y_{t+h-2}' C\left(C_n\left(u_h\right) + \Gamma_{n_{h-1}}\right) Y_{t+h-2} \end{array}\right)\right] \\
=\ & \sum_{(n_1,...,n_h)} \exp\left(A_{(n_1,...,n_h)}\left(u_1,...,u_h\right) + B_{(n_1,...,n_h)}\left(u_1,...,u_h\right)' Y_t + Y_t' C_{(n_1,...,n_h)}\left(u_1,...,u_h\right) Y_t\right)
\end{aligned}$$

we have

$$\begin{aligned}
&\varphi_{t,h+1}\left(u_1,...,u_{h+1}\right) \\
=\ & E_t\left[\exp\left(\sum_{j=1}^{h+1} u_j' X_{t+j}\right)\right] \\
=\ & E_t\left[\exp\left(u_1' X_{t+1}\right) E_{t+1}\left[\exp\left(\sum_{i=1}^{h} u_{i+1}' X_{t+1+i}\right)\right]\right] \\
=\ & E_t\left[\exp\left(u_1' X_{t+1}\right) \varphi_{t+1,h}\left(u_2,...,u_{h+1}\right)\right] \\
=\ & E_t\left[\exp\left(u_1' X_{t+1}\right) \sum_{(n_1,...,n_h)} \exp\left(\begin{array}{c} A_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) \\ +B_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right)' Y_{t+1} + \\ Y_{t+1}' C_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) Y_{t+1} \end{array}\right)\right] \\
=\ & \sum_{(n_1,...,n_h)} E_t\left[\exp\left(\begin{array}{c} A_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) + u_1' X_{t+1} \\ +B_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right)' Y_{t+1} \\ +Y_{t+1}' C_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) Y_{t+1} \end{array}\right)\right] \\
=\ & \sum_{(n_1,...,n_h)} E_t\left[\exp\left(\begin{array}{c} A_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) + u_{1,r}' r_{t+1} \\ + \left(B_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) + u_{1,y}\right)' Y_{t+1} \\ +Y_{t+1}' C_{(n_1,...,n_h)}\left(u_2,...,u_{h+1}\right) Y_{t+1} \end{array}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\varphi_{t,h+1}(u_1,...,u_{h+1}) \\
= & \sum_{(n_1,...,n_h)} \sum_{n=0}^{\bar{n}} E_t \left[\exp \left(\begin{array}{c} A_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \bar{\gamma}_{0,n}(u_{1,r}) + \\ \left(B_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + u_{1,y} + \beta_n\right)' Y_{t+1} \\ +Y_{t+1}' \left(C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_n\right) Y_{t+1} \end{array} \right) \right] \\
= & \sum_{(n_1,...,n_h)} \sum_{n=0}^{\bar{n}} \exp \left(\begin{array}{c} A_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \bar{\gamma}_{0,n}(u_{1,r}) \\ +A\left(B_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + u_{1,y} + \beta_n, C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_n\right) \\ +B\left(B_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + u_{1,y} + \beta_n, C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_n\right)' Y_t \\ +Y_t' C\left(C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_n\right) Y_t \end{array} \right) \\
= & \sum_{(n_1,...,n_{h+1})} \exp \left(\begin{array}{c} A_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1}) + B_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1})' Y_t \\ +Y_t' C_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1}) Y_t \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
&A_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1}) \\
= \; & A_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \bar{\gamma}_{0,n_{h+1}}(u_{1,r}) + \\
& A\left(B_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + u_{1,y} + \beta_{n_{h+1}}, C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_{n_{h+1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&B_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1}) \\
= \; & B\left(B_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + u_{1,y} + \beta_{n_{h+1}}, C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_{n_{h+1}}\right)
\end{aligned}
$$

and

$$
C_{(n_1,...,n_{h+1})}(u_1,...,u_{h+1}) = C\left(C_{(n_1,...,n_h)}(u_2,...,u_{h+1}) + \Gamma_{n_{h+1}}\right)
$$

$$
\varphi_{t,h}(u) \equiv E_t\left[\exp\left(u' \sum_{j=1}^{h} X_{t+j}\right)\right] = \sum_{(n_1,...,n_h)} \exp\left(A_{(n_1,...,n_h)}(u) + B_{(n_1,...,n_h)}(u)' Y_t + Y_t' C_{(n_1,...,n_h)}(u) Y_t\right)
$$

where

$$
\begin{aligned}
A_{(n_1,...,n_{h+1})}(u) &= A_{(n_1,...,n_h)}(u) + \bar{\gamma}_{0,n_{h+1}}(u_r) + A\left(B_{(n_1,...,n_h)}(u) + u_y + \beta_{n_{h+1}}, C_{(n_1,...,n_h)}(u) + \Gamma_{n_{h+1}}\right) \\
B_{(n_1,...,n_{h+1})}(u) &= B\left(B_{(n_1,...,n_h)}(u) + u_y + \beta_{n_{h+1}}, C_{(n_1,...,n_h)}(u) + \Gamma_{n_{h+1}}\right) \\
C_{(n_1,...,n_{h+1})}(u) &= C\left(C_{(n_1,...,n_h)}(u) + \Gamma_{n_{h+1}}\right)
\end{aligned}
$$

B Asset Pricing

B.1 Change Of Measure

let denote the log-SDF by $m_{t+1} = \ln(M_{t+1})$, the pricing kernel is

$$
\xi_{t+1} = \frac{M_{t+1}}{E_t[M_{t+1}]}
$$

so that

$$
\begin{aligned}
M_{t+1} &= \xi_{t+1} E_t[M_{t+1}] \\
&= \xi_{t+1} \exp(-r_t)
\end{aligned}
$$

we will assume that

$$
\xi_{t+1} = \xi_{t+1}^{Y} \xi_{t+1}^{r}
$$

with

$$\xi_{t+1}^{Y}\left(Y_{t+1}\right)=\frac{\exp\left(\lambda_{y,t}'Y_{t+1}\right)}{E_t\left[\exp\left(\lambda_{y,t}'Y_{t+1}\right)\right]}=\exp\left(\lambda_{y,t}'Y_{t+1}-\lambda_{y,t}'\left(\omega+\phi Y_t\right)-\frac{1}{2}\lambda_{y,t}'\Sigma\Sigma'\lambda_{y,t}\right)$$

$$\xi_{t+1}^{r}\left(\bar{\Delta}_{t+1}\right)=\frac{Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}\right]}{P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}\right]}$$

where we define Q similarly to P, which means that

$$Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]=CDF_{t|Y_{t+1}}^{Q}\left(\delta_{n,t}\left(Y_{t+1}\right)\right)-CDF_{t|Y_{t+1}}^{Q}\left(\delta_{n-1,t}\left(Y_{t+1}\right)\right)$$

with

$$CDF_{t|Y_{t+1}}^{Q}\left(\delta_{n,t}\left(Y_{t+1}\right)\right)=\exp\left(\alpha_n^*+\beta_n^{*\prime}Y_{t+1}+Y_{t+1}'\Gamma_n^*Y_{t+1}\right)$$

and

$$\begin{aligned}\alpha_{\bar{n}}^* &= 0,\quad \beta_{\bar{n}}^*=0,\quad \Gamma_{\bar{n}}^*=0\\ &\Gamma_n^*-\Gamma_{n-1}^*\text{ a semi defnite positive matrix}\\ \text{and } \alpha_n^*-\alpha_{n-1}^* &\geq \frac{1}{4}\left(\beta_n^*-\beta_{n-1}^*\right)'\left(\Gamma_n^*-\Gamma_{n-1}^*\right)^{-1}\left(\beta_n^*-\beta_{n-1}^*\right)\end{aligned}$$

to ease the presentation, we will add the following condition

$$\alpha_{-1}^*=-\infty,\quad \Gamma_{-1}^*=-\infty$$

thus

$$\xi_{t+1}^{r}=\frac{\exp\left(\alpha_{\bar{\Delta}_{t+1}}^*+\beta_{\bar{\Delta}_{t+1}}^{*\prime}Y_{t+1}+Y_{t+1}'\Gamma_{\bar{\Delta}_{t+1}}^*Y_{t+1}\right)-\exp\left(\alpha_{\bar{\Delta}_{t+1}-1}^*+\beta_{\bar{\Delta}_{t+1}-1}^{*\prime}Y_{t+1}+Y_{t+1}'\Gamma_{\bar{\Delta}_{t+1}-1}^*Y_{t+1}\right)}{\exp\left(\alpha_{\bar{\Delta}_{t+1}}+\beta_{\bar{\Delta}_{t+1}}'Y_{t+1}+Y_{t+1}'\Gamma_{\bar{\Delta}_{t+1}}Y_{t+1}\right)-\exp\left(\alpha_{\bar{\Delta}_{t+1}-1}+\beta_{\bar{\Delta}_{t+1}-1}'Y_{t+1}+Y_{t+1}'\Gamma_{\bar{\Delta}_{t+1}-1}Y_{t+1}\right)}$$

we have

$$E_t\left[\xi_{t+1}\right]=E_t\left[\xi_{t+1}^{Y}\xi_{t+1}^{r}\right]=E_t\left[\xi_{t+1}^{Y}E_t\left[\xi_{t+1}^{r}|Y_{t+1}\right]\right]$$

$$\begin{aligned}E_t\left[\xi_{t+1}^{r}|Y_{t+1}\right] &= E_t\left[\frac{Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}\right]}{P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}\right]}|Y_{t+1}\right]\\ &= \sum_{n=0}^{n}\frac{Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]}{P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]}P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]\\ &= \sum_{n=0}^{n}Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]\\ &= 1\end{aligned}$$

hence

$$\begin{aligned}E_t\left[\xi_{t+1}\right] &= E_t\left[\xi_{t+1}^{Y}E_t\left[\xi_{t+1}^{r}|Y_{t+1}\right]\right]\\ &= E_t\left[\xi_{t+1}^{Y}\right]=1\end{aligned}$$

thus ξ_{t+1} is a valid pricing kernel

We have

$$
\begin{aligned}
E_t^Q\left[\exp\left(u_r r_{t+1}\right)|Y_{t+1}\right] &= E_t\left[\exp\left(u_r c\bar{\Delta}_{t+1}\right)\frac{\xi_{t+1}}{E_t\left[\xi_{t+1}|Y_{t+1}\right]}|Y_{t+1}\right] = \\
E_t\left[\exp\left(u_r c\bar{\Delta}_{t+1}\right)\frac{\xi_{t+1}}{E_t\left[\xi_{t+1}|Y_{t+1}\right]}|Y_{t+1}\right] &= E_t\left[\exp\left(u_r c\bar{\Delta}_{t+1}\right)\xi_{t+1}^r|Y_{t+1}\right] \\
&= \sum_{n=0}^{\bar{n}}\exp\left(u_r cn\right)\frac{Q_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]}{P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right]}P_{t|Y_{t+1}}\left[\bar{\Delta}_{t+1}=n\right] \\
&= \sum_{n=0}^{\bar{n}}\exp\left(\bar{\gamma}_{0,n}^*\left(u_r\right)+\beta_n^{*\prime}Y_{t+1}+Y_{t+1}'\Gamma_n^*Y_{t+1}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\gamma}_{0,n}^*\left(u_r\right) &= \ln\left(1-\exp\left(u_r c\right)\right)+u_r cn+\alpha_n^* \\
for \quad n &\leq \bar{n}
\end{aligned}
$$

and

$$
\bar{\gamma}_{0,\bar{n}}\left(u_r\right)=u_r c\bar{n}
$$

thus

$$
E_t^Q\left[\exp\left(u_r r_{t+1}\right)|Y_{t+1}\right]=\sum_{n=0}^{\bar{n}}\exp\left(\bar{\gamma}_{0,n}^*\left(u_r\right)+\beta_n^{*\prime}Y_{t+1}+Y_{t+1}'\Gamma_n^*Y_{t+1}\right)
$$

we have

$$
\begin{aligned}
E_t^Q\left[\exp\left(u_y'Y_{t+1}\right)\right] &= E_t\left[\exp\left(u_y'Y_{t+1}\right)\xi_{t+1}^r\xi_{t+1}^Y\right] \\
&= E_t\left[\exp\left(u_y'Y_{t+1}\right)\xi_{t+1}^Y E_t\left[\xi_{t+1}^r|Y_{t+1}\right]\right] \\
&= E_t\left[\exp\left(u_y'Y_{t+1}\right)\xi_{t+1}^Y\right] \\
&= E_t\left[\exp\left(u_y'Y_{t+1}+\lambda_{y,t}'Y_{t+1}-\lambda_{y,t}'\left(\omega+\phi Y_t\right)-\frac{1}{2}\lambda_{y,t}'\Sigma\Sigma'\lambda_{y,t}\right)\right] \\
&= \exp\begin{pmatrix} -\lambda_{y,t}'\left(\omega+\phi Y_t\right)-\frac{1}{2}\lambda_{y,t}'\Sigma\Sigma'\lambda_{y,t}+\left(u_y+\lambda_{y,t}\right)'\left(\omega+\phi Y_t\right) \\ +\frac{1}{2}\left(u_y+\lambda_{y,t}\right)'\Sigma\Sigma'\left(u_y+\lambda_{y,t}\right) \end{pmatrix} \\
&= \exp\left(u_y'\left(\omega+\phi Y_t\right)+\frac{1}{2}u_y'\Sigma\Sigma'u_y+u_y'\Sigma\Sigma'\lambda_{y,t}\right)
\end{aligned}
$$

we choose a linear price of risk as following

$$
\lambda_{y,t}=\lambda_0+\lambda_1 Y_t
$$

which implies that

$$
\begin{aligned}
E_t^Q\left[\exp\left(u_y'Y_{t+1}\right)\right] &= u_y'\left(\omega+\phi Y_t\right)+\frac{1}{2}u_y'\Sigma\Sigma'u_y+u_y'\Sigma\Sigma'\lambda_{y,t} \\
&= u_y'\left(\omega+\phi Y_t\right)+\frac{1}{2}u_y'\Sigma\Sigma'u_y+u_y'\Sigma\Sigma'\left(\lambda_0+\lambda_1 Y_t\right) \\
&= u_y'\left(\omega+\Sigma\Sigma'\lambda_0+\left(\phi+\Sigma\Sigma'\lambda_1\right)Y_t\right)+\frac{1}{2}u_y'\Sigma\Sigma'u_y
\end{aligned}
$$

thus

$$
Y_{t+1}=\omega^*+\phi^*Y_t+\eta_{t+1}^*
$$

where

$$\begin{aligned}\omega^* &= \omega + \Sigma\Sigma'\lambda_0 \\ \phi^* &= \phi + \Sigma\Sigma'\lambda_1 \\ \eta^*_{t+1} &\sim {}^{Q} iidN\left(0, \Sigma\Sigma'\right)\end{aligned}$$

B.2 Bond Prices

Bond price at maturity $h+1$, is

$$D_t(h+1) = E_t^Q\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)\right] = E_t^Q\left[\exp\left(e_1^{\intercal}\sum_{j=0}^{h} X_{t+j}\right)\right]$$

where

$$e_1 = -\left(1, 0, ..., 0\right)^{\intercal}$$

we have

$$e^{r_t} D_t(h+1) = \sum_{(n_1,\ldots,n_h)} \exp\left(a_{(n_1,\ldots,n_h)} + b^{\intercal}_{(n_1,\ldots,n_h)} Y_t + Y_t^{\intercal} c_{(n_1,\ldots,n_h)} Y_t\right)$$

$$\begin{aligned}a_{(n_1,\ldots,n_h)} &= A^*\left(b_{(n_1,\ldots,n_{h-1})} + \beta^*_{n_h}, c_{(n_1,\ldots,n_{h-1})} + \Gamma^*_{n_h}\right) \\ &\quad + a_{(n_1,\ldots,n_{h-1})} + \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n_h<\bar{n}]}} - cn_h + \alpha^*_{n_h} \\ b_{(n_1,\ldots,n_h)} &= B^*\left(b_{(n_1,\ldots,n_{h-1})} + \beta^*_{n_h}, c_{(n_1,\ldots,n_{h-1})} + \Gamma^*_{n_h}\right) \\ c_{(n_1,\ldots,n_h)} &= C^*\left(c_{(n_1,\ldots,n_{h-1})} + \Gamma^*_{n_h}\right)\end{aligned}$$

where $1_{[n_h<\bar{n}]}$ is the indicator function that take 1 if $n_h < \bar{n}$ and 0 elsewhere

The initial conditions are

$$\begin{aligned}a_n &= \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n<\bar{n}]}} - cn + \alpha^*_n + A\left(\beta^*_n, \Gamma^*_n\right) \\ b_n &= B\left(\beta^*_n, \Gamma^*_n\right) \\ c_n &= C\left(\Gamma^*_n\right)\end{aligned}$$

How to compute coefficient $a_{(n_1,\ldots,n_h)}$, $b_{(n_1,\ldots,n_h)}$ and $c_{(n_1,\ldots,n_h)}$ in practice?

1. Start by initial value

$$\begin{aligned}a_{n_1} &= \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n_1<\bar{n}]}} - cn_1 + \alpha^*_{n_1} + A\left(\beta^*_{n_1}, \Gamma^*_{n_1}\right) \\ b_{n_1} &= B\left(\beta^*_{n_1}, \Gamma^*_{n_1}\right) \\ c_{n_1} &= C\left(\Gamma^*_{n_1}\right)\end{aligned}$$

2.

$$\begin{aligned}a_{(n_1,n_2)} &= A^*\left(b_{n_1} + \beta^*_{n_2}, c_{n_1} + \Gamma^*_{n_2}\right) \\ &\quad + a_{n_1} + \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n_2=\bar{n}]}} - cn_2 + \alpha^*_{n_2} \\ b_{(n_1,n_2)} &= B^*\left(b_{n_1} + \beta^*_{n_2}, c_{n_1} + \Gamma^*_{n_2}\right) \\ c_{(n_1,n_2)} &= C^*\left(c_{n_1} + \Gamma^*_{n_2}\right)\end{aligned}$$

3.

$$\begin{aligned} a_{(n_1,n_2,n_3)} &= A^*\left(b_{(n_1,n_2)} + \beta^*_{n_3}, c_{(n_1,n_2)} + \Gamma^*_{n_3}\right) \\ &\quad + a_{(n_1,n_2)} + \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n_3=\bar{n}]}} - cn_3 + \alpha^*_{n_3} \\ b_{(n_1,n_2,n_3)} &= B^*\left(b_{(n_1,n_2)} + \beta^*_{n_3}, c_{(n_1,n_2)} + \Gamma^*_{n_3}\right) \\ c_{(n_1,n_2,n_3)} &= C^*\left(c_{(n_1,n_2)} + \Gamma^*_{n_3}\right) \end{aligned}$$

4. $\vdots$

5.

$$\begin{aligned} a_{(n_1,\dots,n_h)} &= A^*\left(b_{(n_1,\dots,n_{h-1})} + \beta^*_{n_h}, c_{(n_1,\dots,n_{h-1})} + \Gamma^*_{n_h}\right) \\ &\quad + a_{(n_1,\dots,n_{h-1})} + \ln\left(1 - \exp\left(-c\right)\right)^{1_{[n_h=\bar{n}]}} - cn_h + \alpha^*_{n_h} \\ b_{(n_1,\dots,n_h)} &= B^*\left(b_{(n_1,\dots,n_{h-1})} + \beta^*_{n_h}, c_{(n_1,\dots,n_{h-1})} + \Gamma^*_{n_h}\right) \\ c_{(n_1,\dots,n_h)} &= C^*\left(c_{(n_1,\dots,n_{h-1})} + \Gamma^*_{n_h}\right) \end{aligned}$$

B.3 Option Prices

Option price that pays $(r_{t+h+1} - nc)^+$ maturity $h+1$ where $n = 0, 1..., \bar{n}$, is

$$\begin{aligned} OP_t(h+1) &= E^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)(r_{t+h+1} - nc)^+\right] \\ &= E^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[(r_{t+h+1} - nc)^+\right]\right] \\ &= E^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[(r_{t+h+1} - nc)\,1_{\left[\bar{\Delta}_{t+h+1}>n\right]}\right]\right] \\ &= cE^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[\bar{\Delta}_{t+h+1}1_{\left[\bar{\Delta}_{t+h+1}>n\right]}\right]\right] \\ &\quad - ncE^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[1_{\left[\bar{\Delta}_{t+h+1}>n\right]}\right]\right] \end{aligned}$$

$$\begin{aligned} OP_t(h+1) &= cE^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[\bar{\Delta}_{t+h+1}\right]\right] \\ &\quad - cE^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)E_{t+h}\left[\bar{\Delta}_{t+h+1}1_{\left[\bar{\Delta}_{t+h+1}\le n\right]}\right]\right] \\ &\quad + ncE^Q_t\left[\exp\left(\begin{array}{c} \alpha_n + A\left(\beta_n, \Gamma_n\right) + B\left(\beta_n, \Gamma_n\right)' Y_{t+h} \\ + Y'_{t+h} C\left(\Gamma_n\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j} \end{array}\right)\right] \\ &\quad - ncE^Q_t\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)\right] \end{aligned}$$

$$
\begin{aligned}
OP_t(h+1) &= c\left(\bar{n}+1\right)E_t^Q\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)\right] - c\sum_{m=0}^{\bar{n}} E_t^Q\left[\exp\left(\begin{array}{c}\alpha_m + A\left(\beta_m,\Gamma_m\right) + B\left(\beta_m,\Gamma_m\right)' Y_{t+h} \\ +Y_{t+h}' C\left(\Gamma_m\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\end{array}\right)\right] \\
&-c\left(n+1\right)E_t^Q\left[\exp\left(\begin{array}{c}\alpha_n + A\left(\beta_n,\Gamma_n\right) + B\left(\beta_n,\Gamma_n\right)' Y_{t+h} \\ +Y_{t+h}' C\left(\Gamma_n\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\end{array}\right)\right] \\
&+c\sum_{m=0}^{n} E_t^Q\left[\exp\left(\begin{array}{c}\alpha_m + A\left(\beta_m,\Gamma_m\right) + B\left(\beta_m,\Gamma_m\right)' Y_{t+h} \\ +Y_{t+h}' C\left(\Gamma_m\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\end{array}\right)\right] \\
&+ncE_t^Q\left[\exp\left(\begin{array}{c}\alpha_n + A\left(\beta_n,\Gamma_n\right) + B\left(\beta_n,\Gamma_n\right)' Y_{t+h} \\ +Y_{t+h}' C\left(\Gamma_n\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\end{array}\right)\right] - ncE_t^Q\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
OP_t(h+1) &= c\left(\bar{n}-n+1\right)E_t^Q\left[\exp\left(-\sum_{j=0}^{h} r_{t+j}\right)\right] \\
&\quad -c\sum_{m=n}^{\bar{n}} E_t^Q\left[\exp\left(\begin{array}{c}\alpha_m + A\left(\beta_m,\Gamma_m\right) + B\left(\beta_m,\Gamma_m\right)' Y_{t+h} \\ +Y_{t+h}' C\left(\Gamma_m\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\end{array}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& E_t^Q\left[\exp\left(\alpha_m + A\left(\beta_m,\Gamma_m\right) + B\left(\beta_m,\Gamma_m\right)' Y_{t+h} + Y_{t+h}' C\left(\Gamma_m\right) Y_{t+h} - \sum_{j=0}^{h} r_{t+j}\right)\right] \\
= & \sum_{k=0}^{\bar{n}} E_t^Q\left[\exp\left(\begin{array}{c}\bar{\gamma}_{0,k}\left(-1\right) + \alpha_m + A\left(\beta_m,\Gamma_m\right) + \left(\beta_k + B\left(\beta_m,\Gamma_m\right)\right)' Y_{t+h} \\ +Y_{t+h}'\left(\Gamma_k + C\left(\Gamma_m\right)\right) Y_{t+h} - \sum_{j=0}^{h-1} r_{t+j}\end{array}\right)\right] \\
= & \sum_{k=0}^{\bar{n}} E_t^Q\left[\exp\left(\begin{array}{c}\bar{\gamma}_{0,k}\left(-1\right) + \alpha_m + A\left(\beta_m,\Gamma_m\right) \\ +A\left(\beta_k + B\left(\beta_m,\Gamma_m\right),\Gamma_k + C\left(\Gamma_m\right)\right) \\ +B\left(\beta_k + B\left(\beta_m,\Gamma_m\right),\Gamma_k + C\left(\Gamma_m\right)\right)' Y_{t+h-1} \\ +Y_{t+h-1}' C\left(\Gamma_k + C\left(\Gamma_m\right)\right) Y_{t+h-1} - \sum_{j=0}^{h-1} r_{t+j}\end{array}\right)\right] \\
= & \sum_{(n_1,\ldots,n_h)} \exp\left(\alpha_{(n_1,\ldots,n_h)}^m + \beta_{(n_1,\ldots,n_h)}^{m\prime} Y_t + Y_t' \Gamma_{(n_1,\ldots,n_h)}^m Y_t - r_t\right)
\end{aligned}
$$

$$
\begin{aligned}
& E_t^Q\left[\exp\left(\alpha_m + A(\beta_m,\Gamma_m) + B(\beta_m,\Gamma_m)' Y_{t+h+1} + Y'_{t+h+1} C(\Gamma_m) Y_{t+h+1} - \sum_{j=0}^{h+1} r_{t+j}\right)\right] \\
= & \; E_t^Q\left[\sum_{(n_1,\ldots,n_h)} \exp\left(-r_t + \alpha^m_{(n_1,\ldots,n_h)} + \beta^{m\prime}_{(n_1,\ldots,n_h)} Y_{t+1} + Y'_{t+1}\Gamma^m_{(n_1,\ldots,n_h)} Y_{t+1} - r_{t+1}\right)\right] \\
= & \sum_{(n_1,\ldots,n_h)} E_t^Q\left[\exp\left(-r_t + \alpha^m_{(n_1,\ldots,n_h)} + \beta^{m\prime}_{(n_1,\ldots,n_h)} Y_{t+1} + Y'_{t+1}\Gamma^m_{(n_1,\ldots,n_h)} Y_{t+1} - r_{t+1}\right)\right] \\
= & \sum_{(n_1,\ldots,n_h)} \sum_{n_{h+1}} E_t^Q\left[\exp\left(\begin{array}{c} -r_t + \alpha^m_{(n_1,\ldots,n_h)} + \beta^{m\prime}_{(n_1,\ldots,n_h)} Y_{t+1} \\ +Y'_{t+1}\Gamma^m_{(n_1,\ldots,n_h)} Y_{t+1} + \bar{\gamma}_{0,n_{h+1}}(-1) + \beta'_{n_{h+1}} Y_{t+1} + Y'_{t+1}\Gamma_{n_{h+1}} Y_{t+1} \end{array}\right)\right] \\
= & \sum_{(n_1,\ldots,n_h,n_{h+1})} \exp\left(\begin{array}{c} -r_t + \alpha^m_{(n_1,\ldots,n_h)} + \bar{\gamma}_{0,n_{h+1}}(-1) + A\left(\beta^m_{(n_1,\ldots,n_h)} + \beta_{n_{h+1}}, \Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right) \\ +B\left(\beta^m_{(n_1,\ldots,n_h)} + \beta_{n_{h+1}}, \Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right)' Y_t + Y'_t C\left(\Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right) Y_t \end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
\alpha^m_{(n_1,\ldots,n_{h+1})} &= \alpha^m_{(n_1,\ldots,n_h)} + \ln(1-\exp(-c))^{1_{(n_{h+1}<n)}} - cn_{h+1} + \alpha_{n_{h+1}} \\
&\quad + A\left(\beta^m_{(n_1,\ldots,n_h)} + \beta_{n_{h+1}}, \Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right) \\
\beta^m_{(n_1,\ldots,n_{h+1})} &= B\left(\beta^m_{(n_1,\ldots,n_h)} + \beta_{n_{h+1}}, \Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right) \\
\Gamma^m_{(n_1,\ldots,n_{h+1})} &= C\left(\Gamma^m_{(n_1,\ldots,n_h)} + \Gamma_{n_{h+1}}\right)
\end{aligned}
$$

with the following initial

$$
\begin{aligned}
\alpha^m_{n_1} &= \alpha_m + \alpha_{n_1} + A(\beta_m,\Gamma_m) + A(B(\beta_m,\Gamma_m) + \beta_{n_1}, C(\Gamma_m) + \Gamma_{n_1}) + \ln(1-\exp(-c))^{1_{(n_1<n)}} - cn_1 \\
\beta^m_{n_1} &= B(B(\beta_m,\Gamma_m) + \beta_{n_1}, C(\Gamma_m) + \Gamma_{n_1}) \\
\Gamma^m_{n_1} &= C(C(\Gamma_m) + \Gamma_{n_1})
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{e^{r_t} OP_t(h+1)}{c} &= (\bar{n} - n + 1) e^{r_t} B_t(h+1) - \sum_{(n_1,\ldots,n_h)} \sum_{m=n}^{\bar{n}} \exp\left(\alpha^m_{(n_1,\ldots,n_h)} + \beta^{m\prime}_{(n_1,\ldots,n_h)} Y_t + Y'_t \Gamma^m_{(n_1,\ldots,n_h)} Y_t\right) \\
&= (\bar{n} - n + 1) e^{r_t} B_t(h+1) - \sum_{(n_1,\ldots,n_h)} \sum_{m=n}^{\bar{n}} \exp\left(a_{(m,n_1,\ldots,n_h)} + b'_{(m,n_1,\ldots,n_h)} Y_t + Y'_t c_{(m,n_1,\ldots,n_h)} Y_t\right) \\
&= \sum_{(n_1,\ldots,n_h)} \sum_{m=n}^{\bar{n}} \left[\begin{array}{c} \exp\left(a_{(n_1,\ldots,n_h)} + b^\intercal_{(n_1,\ldots,n_h)} Y_t + Y_t^\intercal c_{(n_1,\ldots,n_h)} Y_t\right) \\ -\exp\left(a_{(m,n_1,\ldots,n_h)} + b^\intercal_{(m,n_1,\ldots,n_h)} Y_t + Y_t^\intercal c_{(m,n_1,\ldots,n_h)} Y_t\right) \end{array}\right]
\end{aligned}
$$